Fatih University — Department of Mathematics

MATH 113 — Summer 2004

Midterm Exam I | Time: 75 minutes

Name ....: ....

Student No ..: ....

Department .: ....

Questions

(20 pnts.) **1. Find all intercepts of the function $f(x) = e^x(x^2 - 2x - 7)$.**

For $y-$intercept: $f(0) = e^0(0^2 - 20 - 7) = 1(-7) = -7,$ $(0, -7)$

For $x-$intercept: $f(x) = 0 \quad \Rightarrow \quad e^x(x^2 - 2x - 7) = 0$

Either $e^x = 0$ or $x^2 - 2x - 7 = 0$. Since an exponential function never becomes zero then

$$x^2 - 2x - 7 = 0 \quad \Rightarrow \quad x^2 - 2(1)x + 1 - 1 - 7 = 0 \quad \Rightarrow \quad x^2 - 2(1)x + 1 - 8 = 0$$

$$(x-1)^2 - 8 = 0 \quad \Rightarrow \quad (x-1)^2 = 8 \quad \Rightarrow \quad x - 1 = \pm\sqrt{8}$$

There are two $x-$intercepts $x_1 = 1 + \sqrt{8}$ and $x_2 = 1 - \sqrt{8}$

$(1 + \sqrt{8}, 0)$ and $(1 - \sqrt{8}, 0)$

(20 pnts.) **2. Find the equation of a line passing through a point $(3, 4)$ and perpendicular to the line $x + y = 5$.**

The equation of a line passing through a point is given by

$$y - y_1 = m(x - x_1) \qquad \text{where } (x_1, y_1) = (3, 4) \text{ and the slope is } m = -\frac{1}{m_1}$$

then

$$x + y = 5 \quad \Rightarrow \quad y = -x + 5 \qquad \text{so the slope } m_1 = -1$$

$$y - 4 = -\frac{1}{-1}(x - 3) \quad \Rightarrow \quad y - 4 = x - 3 \quad \Rightarrow \quad y = x - 3 + 4$$

The equation of the line is $y = x + 1$

(20 pnts.) 3. Find the limit $\displaystyle \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+9}-3} = ?$

The limit $\displaystyle \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+9}-3} = \frac{0^2}{\sqrt{0^2+9}-3} = \frac{0}{0}$

We should use a special case which means multiply the denominator by the conjugate of it

$$\lim_{x\to 0} \frac{x^2(\sqrt{x^2+9}+3)}{(\sqrt{x^2+9}-3)(\sqrt{x^2+9}+3)} = \lim_{x\to 0} \frac{x^2(\sqrt{x^2+9}+3)}{x^2+9-3^2} = \lim_{x\to 0} \frac{\cancel{x}^2(\sqrt{x^2+9}+3)}{\cancel{x}^2}$$

$$= \lim_{x\to 0} \left(\sqrt{x^2+9}+3\right) = \sqrt{0^2+9}+3 = 3+3 = 6$$

As a result, $\displaystyle \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+9}-3} = 6$

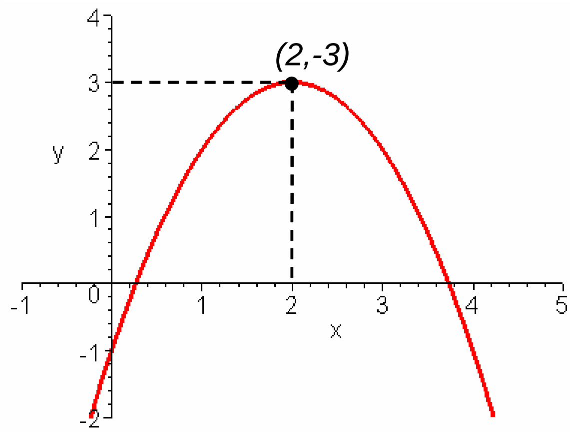
(20 pnts.) 4. Graph the given function $f(x) = -x^2+4x-1.$

This is a graph of a parabola but it is shifted and reflected. First, we should transform the quadratic form into $f(x) = a(x-h)^2+k$.

$$f(x) = -x^2+4x-1 = -(x^2-4x+1) = -(x^2-2(2)x+4-4+1) = -\left((x-2)^2-3\right).$$

$f(x) = -\left((x-2)^2-3\right)$ where $h = 2$ and $k = -3$ so the vertex point is $(2, -3)$

Now, we will shift the vertex of the graph of original parabola x^2 from the origin to $(2, -3)$ and reflected because of the negative sign in front of the function. So the graph should be like that



(20 pnts.) **5. Solve the logarithmic functions** $\log_b x = \frac{2}{3}\log_b 8 - \frac{3}{2}\log_b 4 + \log_b 8$ **for x.**

$$\Rightarrow \quad \log_b x = \frac{2}{3}\log_b 8 - \frac{3}{2}\log_b 4 + \log_b 8$$

$$\Rightarrow \quad \log_b x = \log_b (8)^{\frac{2}{3}} - \log_b (4)^{\frac{3}{2}} + \log_b 8$$

$$\Rightarrow \quad \log_b x = \log_b (2^3)^{\frac{2}{3}} - \log_b (2^2)^{\frac{3}{2}} + \log_b 8$$

$$\Rightarrow \quad \log_b x = \log_b (2)^2 - \log_b (2)^3 + \log_b 8$$

$$\Rightarrow \quad \log_b x = \log_b 4 - \cancel{\log_b 8} + \cancel{\log_b 8}$$

$$\Rightarrow \quad \log_b x = \log_b 4 \quad \Rightarrow \quad x = 4$$

(20 pnts.) **6. Does the following limit exist? Show your answer analytically.** $\lim_{x \to -1} \frac{x-4}{x+1} = ?$

To show the existence of a limit we have to show the following identity

$$\lim_{x \to -1^-} \frac{x-4}{x+1} \stackrel{?}{=} \lim_{x \to -1^+} \frac{x-4}{x+1}$$

$$\lim_{x \to -1^-} \frac{x-4}{x+1} = \frac{-1.0000\cdots0001 - 4}{-1.0000\cdots0001 + 1} = \frac{-5.0000\cdots0001}{-0.0000\cdots0001}$$

$$= \frac{-5.0000\cdots0001}{-\text{Too small number}} = +\text{Too big number} = +\infty$$

$$\lim_{x \to -1^+} \frac{x-4}{x+1} = \frac{-0.9999\cdots9999 - 4}{-0.9999\cdots9999 + 1} = \frac{-4.9999\cdots9999}{+0.0000\cdots0001}$$

$$= \frac{-4.9999\cdots9999}{+\text{Too small number}} = -\text{Too big number} = -\infty$$

Since the identity

$\lim_{x \to -1^-} \frac{x-4}{x+1} = \lim_{x \to -1^+} \frac{x-4}{x+1}$ is not satisfied then the limit $\lim_{x \to -1} \frac{x-4}{x+1}$ does not exist.

(20 pnts.) 7. In which interval the function $\boldsymbol{f(x) = \dfrac{x^3 - 6x^2 + 5x}{x^2 - x}}$ is continuous?

Function $f(x)$ is continuous only out of the singular points in which function is undefined. These points can be found for rational functions by setting the denominator equal to zero such that

$$x^2 - x = x(x-1) = 0 \qquad \Rightarrow \qquad x = 0 \text{ and } \qquad x = 1$$

The function $f(x)$ is continuous in the following intervals

In Interval Notation: $(-\infty, 0)$ and $(0,1)$ and $(1,\infty)$

OR

In Inequality Notation: $-\infty < x < 0$and $0 < x < 1$ and $1 < x < \infty$

(20 pnts.) 8. Find all asymptotes of function $\boldsymbol{f(x) = \dfrac{x^3 - 4x^2 + 3x}{x^2 + x - 2}}$

A) Vertical asymptote can be found by setting the denominator equal to zero. In other words, the points in which function is undefined are called as vertical asymptote.

$$x^2 + x - 2 = (x+2)(x-1) = 0 \qquad \Rightarrow \qquad x = -2 \text{ and } \qquad x = 1$$

At $x = -2$ and $x = 1$, there are two vertical asymptotes.

B) Horizontal asymptotes can be found by examining the limit of the function for large x values such as

$$\lim_{x\to\pm\infty} \frac{x^3 - 4x^2 + 3x}{x^2 + x - 2} = \lim_{x\to\pm\infty} \frac{x(x-3)(x-1)}{(x+2)(x-1)} = \lim_{x\to\pm\infty} \frac{x(x-3)}{(x+2)} = \pm\infty$$

Since the limit equals to infinity then there is no horizontal asymptotes.

C) There is one more type of asymptotes that is called an oblique asymptote if the degree of numerator is one more than the degree of denominator. To show the oblique asymptote the numerator should be divided by the denominator.

$$\frac{x^3 - 4x^2 + 3x}{x^2 + x - 2} = \frac{x(x-3)(x-1)}{(x+2)(x-1)} = \frac{x(x-3)}{(x+2)} = x - 5 + \frac{10}{x+2}$$

Then, $x - 5$ is the oblique asymptotes and it is given as $y = x - 5$.