

Math114 Final Exam (17/01/2007)

Questions

Name&Number

1) Express the limit as definite integral

$$\lim_{\|P\| \rightarrow 0} \sum_{k=1}^n \left(\frac{1}{c_k^2}\right) \Delta \mathbf{x}_k \quad \text{where } P \text{ is a partition of } [1,4] \text{ [10P]}$$

Solution: $\int_1^4 \frac{1}{x^2} dx$

2) Evaluate the integrals [60P]

a) $\int_1^2 x \ln x dx$ b) $\int e^{\sqrt{3s+9}} ds$ (hint: set $3s+9=t^2$) c) $\int \frac{dx}{(x-1)(x^2+1)^2}$ d) $\int \cos^3 x dx$

Solution: a)

$$\int_1^2 x \ln x dx = 2 \ln 2 - \int_1^2 \frac{1}{2} x dx = 2 \ln 2 - \frac{3}{4}$$

b)

$$\int e^{\sqrt{3s+9}} ds = \int \frac{2}{3} t e^t dt = \frac{2}{3} t e^t - \frac{2}{3} e^t + c \text{ since } 3s+9 = t^2 \rightarrow ds = 2tdt/3$$

c)

$$\begin{aligned} \int \frac{dx}{(x-1)(x^2+1)^2} &= \int \frac{A}{x-1} dx + \int \frac{Bx+C}{x^2+1} dx + \int \frac{Dx+F}{(x^2+1)^2} dx \\ \frac{1}{(x-1)(x^2+1)^2} &= \frac{1}{4(x-1)} + \frac{1}{(x^2+1)^2} \left(-\frac{1}{2}x - \frac{1}{2}\right) + \frac{1}{x^2+1} \left(-\frac{1}{4}x - \frac{1}{4}\right) \\ \int \frac{dx}{(x-1)(x^2+1)^2} &= c - \frac{1}{2} \arctan x + \frac{1}{4} \ln(x-1) - \frac{1}{8} \ln(x^2+1) + \frac{1}{4x^2+4} - \frac{x}{4x^2+4} \end{aligned}$$

d)

$$\begin{aligned} \int \cos^3 x dx &= \int \cos^2 x \cos x dx = \int (1 - \sin^2 x) \cos x dx \text{ Let's set } \sin x = u \\ &\Rightarrow \int \cos^3 x dx = \int (1 - u^2) du = u - \frac{1}{3} u^3 + c = \sin x - \frac{1}{3} \sin^3 x + c \end{aligned}$$

3) a) Find the area of the region enclosed by the curve and line b) and valume of the region bounded by the given curve revolved about x axis c) and length of the curve on $0 \leq x \leq 4$ [45P]

$$\sqrt{x} + y = 3, \quad x = 0, \quad y = 0$$

Solution: a) Area = $\int_0^4 (3 - \sqrt{x}) dx = 3x - \frac{2}{3}x^{\frac{3}{2}} \Big|_0^4 = \frac{20}{3}$

b) Volume = $\int_0^4 \pi(3 - \sqrt{x})^2 dx = 9\pi x + \frac{1}{2}\pi x^2 - 4\pi x^{\frac{3}{2}} \Big|_0^4 = 12\pi$

c) Length = $\int_0^4 \sqrt{1 + [(3 - \sqrt{x})']^2} dx = \int_0^4 \sqrt{1 + [\frac{1}{2\sqrt{x}}]^2} dx = \int_0^4 \sqrt{1 + \frac{1}{4x}} dx$

4) Find dy/dx if $y = \int_0^{e^{2x}} \sinh(t^2) dt$ and write it in terms of x only [10P]

$$(hint : \sinh x = \frac{e^x + e^{-x}}{2})$$

Solution: $y = \int_0^{e^{2x}} \sinh(t^2) dt \Rightarrow y' = 2e^{2x} \sinh e^{4x} = 2e^{2x} \frac{e^{e^{4x}} + e^{-e^{4x}}}{2}$ it follows

from Fundamental Theorem of Calculus.

5) Evaluate the Improper Integrals [10P]

$$\begin{aligned} a) \int_0^3 \frac{dx}{x-2} &= \int_0^2 \frac{dx}{x-2} + \int_2^3 \frac{dx}{x-2} \Rightarrow \lim_{b \rightarrow 2^-} \int_0^b \frac{dx}{x-2} + \lim_{b \rightarrow 2^+} \int_b^3 \frac{dx}{x-2} \\ \int_0^3 \frac{dx}{x-2} &= \lim_{b \rightarrow 2^-} \ln |b-2| - \ln 2 - \lim_{b \rightarrow 2^+} \ln |b-2| = \lim_{\epsilon \rightarrow 0} \ln |2-\epsilon-2| - \lim_{\epsilon \rightarrow 0} \ln |2+\epsilon-2| - \ln 2 \\ \int_0^3 \frac{dx}{x-2} &= \lim_{\epsilon \rightarrow 0} \ln |-\epsilon| - \lim_{\epsilon \rightarrow 0} \ln |\epsilon| - \ln 2 = \lim_{\epsilon \rightarrow 0} \ln \frac{|-\epsilon|}{|\epsilon|} - \ln 2 = -\ln 2 \end{aligned}$$

6) Please test for convergence [20P]

$$a) \int_{-\infty}^{+\infty} \frac{dx}{e^x + e^{-x}} \quad b) \int_1^{\infty} \frac{\sin x^2 dx}{x^2 + 1}$$

Solution: a) $\int_{-\infty}^{+\infty} \frac{dx}{e^x + e^{-x}} = 2 \int_0^{+\infty} \frac{dx}{e^x + e^{-x}}$ since $\frac{1}{e^x + e^{-x}}$ is even function
 $\Rightarrow \int_{-\infty}^{+\infty} \frac{dx}{e^x + e^{-x}} < 2 \int_0^{+\infty} \frac{dx}{e^x}$ since $e^{-x} > 0$ and $\int_0^{+\infty} \frac{dx}{e^x} = 1 \Rightarrow \int_{-\infty}^{+\infty} \frac{dx}{e^x + e^{-x}}$ is convergent.

b) $\int_1^{\infty} \frac{\sin x^2 dx}{x^2 + 1} \leq \int_1^{\infty} \frac{dx}{x^2 + 1}$ since $\sin x^2 \leq 1$ & $\int_1^{\infty} \frac{dx}{x^2 + 1} = \frac{1}{4}\pi \Rightarrow \int_1^{\infty} \frac{\sin x^2 dx}{x^2 + 1}$ is convergent.

7) Express $1, 12\overline{23}$ as the ratio of two integers by using power series expansion method [10P]

$$1, 12\overline{23} = 1 + \frac{12}{100} + \frac{23}{10^3} + \frac{23}{10^4} + \dots + \frac{23}{10^{n-1}} + \dots = \frac{112}{100} + \frac{23}{10^3} (1 + \frac{1}{10} + \frac{1}{10^2} + \dots) = \frac{112}{100} + \frac{23}{10^3} (\frac{1}{1 - \frac{1}{10}})$$

$$\Rightarrow 1,12232323..... = \frac{112}{100} + \frac{23}{10^3} \left(\frac{1}{1-\frac{1}{10}} \right) = \frac{112}{100} + \frac{23}{10^3} \frac{10}{9} = \frac{112}{100} + \frac{23}{90} = \frac{619}{450}$$

8) Find the values of x which the given geometric series is convergent [10P]

$$a) \sum_{n=0}^{\infty} (-1)^n (2x-1)^n$$

Solution: $\sum_{n=0}^{\infty} (-1)^n (2x-1)^n = \sum_{n=0}^{\infty} [(-1)(2x-1)]^n = \sum_{n=0}^{\infty} [(1-2x)]^n$ convergent
 $\Leftrightarrow |1-2x| < 1 \Leftrightarrow -1 < 2x-1 < 1 \Leftrightarrow 0 < 2x < 2 \Leftrightarrow 0 < x < 1$

9) Determine Why convergent or divergent ? [30P]

$$a) \sum_{n=0}^{\infty} \frac{n}{2n+1} \quad b) \sum_{n=0}^{\infty} \frac{1}{\sqrt{3n}} \quad c) \sum_{n=1}^{\infty} \frac{e^x}{1+e^{2x}} \text{ (hint: use integral test \& set } e^x = y \text{)}$$

Solution: a) $\sum_{n=0}^{\infty} \frac{n}{2n+1}$ divergent since $\lim_{n \rightarrow \infty} \frac{n}{2n+1} = \frac{1}{2}$

b) $\sum_{n=0}^{\infty} \frac{1}{\sqrt{3n}}$ divergent since $\sum_{n=0}^{\infty} \frac{1}{\sqrt{3n}} > \sum_{n=0}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=0}^{\infty} \frac{1}{n}$ harmonic series and divergent

c) $\sum_{n=1}^{\infty} \frac{e^x}{1+e^{2x}}$ & $\int_1^{\infty} \frac{e^x}{1+e^{2x}} dx$ both diverges or converges together. Since $\int_1^{\infty} \frac{e^x}{1+e^{2x}} dx = \int_e^{\infty} \frac{1}{y^2+1} dy = \frac{1}{2}\pi - \arctan e \Rightarrow \sum_{n=1}^{\infty} \frac{e^x}{1+e^{2x}}$ convergent