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Portfolio Selection by Using Time Varying Covariance Matrices

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Abstract. Markowitz mean-variance portfolio theory is one of the most widely used approaches in portfolio selection. Since Markowitz portfolio theory uses equally weighted data, it does not exhibit the current state of the market. It reflects market conditions which are no longer valid by assigning equal weights to the most recent and the most distant observations. To express the dynamic structure of the market, one can use exponentially weighted variances. Exponentially weighted data gives greater weight to the most recent observation. Thus, current market conditions are taken into consideration more accurately. Additionally, to handle the dynamic structure of the volatility in the market, generalised autoregressive conditionally heteroscedastic models can be employed to estimate the covariance matrix.

This paper presents the use of exponentially weighted moving averages and generalised autoregressive conditional heteroscedasticity techniques in portfolio selection. The security variances and the covariance term between each security are calculated using exponentially weighted and GARCH(p,q) schemes. In addition, equally weighted, exponentially weighted and GARCH(p,q) schemes are used for security returns from the XU030 index and portfolio risk parameters at a certain level of expected return are compared. Deviations from the Markowitz mean-variance portfolio theory are investigated.

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Keywords: Markowitz Portfolio Theory, Mean-variance portfolio selection, Exponentially weighted moving averages, Generalised Auto-Regressive Conditional Heteroscedasticity

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1. Introduction

Harry Markowitz (1952) revolutionized the field of portfolio theory with his pioneering work which gained him a share of the 1990 Nobel Prize for Economics. The most important contribution of Markowitz was the use of standard deviation of returns as a measure of risk. He determined the optimum allocation of wealth of an investor by maximizing the expected return at a certain level of risk or minimizing the risk at a certain level of expected return.

Both models are an ex-ante model of portfolio analysis. In other words, to use Markowitz's approach, an estimate of expected returns, variances and covariances must be calculated. The typical procedure used to obtain the inputs is to assign historical ex-post values. Using past returns, one can easily calculate these parameters by giving an equal weight to each period observed from the market. However, using the ex-post data in estimating the future ex-ante parameters of the portfolio can produce disappointing results. One of the main reasons for this failure stems from the estimation of risk. Markowitz portfolio theory uses equally weighted data. Therefore, it does not exhibit the dynamic structure of the market. One way to reduce the estimation errors is to use exponentially weighted returns and variances. Exponentially weighted data gives greater weight to the most recent observation. Thus, current market conditions are taken into consideration more accurately.

Related studies are made in the equity market, weighting recent observations more heavily than older observations by using exponentially moving average (EWMA) techniques or autoregressive conditional heteroscedastic (ARCH) models and generalised autoregressive conditional heteroscedastic (GARCH) models. Akgiray (1989) uses different smoothing parameters of 0.76 to 0.89 and shows that using EWMA techniques are more powerful than the equally weighted scheme. Additionally, Vasiellis and Meade (1996) found that; mean, variance and covariance parameters become unstable during market shocks. Therefore, as market pattern changes dramatically, selected portfolios become unreliable. In addition to this, Tse (1991) compared the forecasting volatility of GARCH and EWMA techniques and found that GARCH forecasts are slower to react to the changes in volatility. Ray and Nawrocki (1996) developed time weighted portfolio optimization by assigning linear weights to past observations, a distributed lag approach. According to their approach, the data at time t is weighted by $t \ / \ \Sigma T$, where T is the number of observations taken into consideration. They used 36 monthly data thus, the weight 36/666 = 5.4% is assigned to the most recent data whereas 1/666 = 0.15% is assigned to the oldest one. Basically when T is equal to 36, the weight attached to the most recent observation is 36 times bigger than the most distant data point. It is obvious that for the equally weighted scheme, in contrast to the exponentially weighted one, the weight of 1/36 = 2.8% is used for each observation.

It is clear from the recent studies that, most of the financial academic literature has focused on modelling the covariance of financial security returns. Besides the academic literature, many industrial contributions have been made such as RiskMetrics (1996). J.P. Morgan and Reuters introduced RiskMetrics methodology for determination and diversification of the market risk of portfolios, a method that soon became popular.

This paper compares alternative ways of calculating the covariance matrix such as EWMA and GARCH. The covariance matrix obtained is used as an input for the Markowitz theory and the effects on risk parameters are investigated. In order to compare the results obtained from the Markowitz approach, the covariance matrix is calculated with an equally weighted framework as well. Daily observations within the period 09.08.2005-30.12.2005 are taken into consideration and returns of fifteen securities from the XU030 index are used as input. There is no particular reason for selecting the period and the specified securities. The main idea is to compare different techniques using the same input data.

The next section outlines the portfolio selection procedure introduced by Markowitz (1952) whereas the third section describes the exponentially weighted scheme. The use of generalised autoregressive conditional heteroscedasticity on modelling volatility is explained in the fourth section. The fifth section outlines the data and methodology used in the paper. The results are presented and discussed in section five and finally, the sixth section summarises conclusion and gives directions for future research.

2. Portfolio Optimization

Modern portfolio theory is based on the idea that investors seek high investment returns and wish to minimize their risk. Expecting higher returns with a lower level of risk is contradictory; therefore, constructing a portfolio requires a trade off between risk and return. Thus, investors must allocate their wealth among different securities. This is known as diversification. Mean-variance optimization developed by Markowitz (1952) can be used in order to determine how an investor allocates his wealth among securities.

The proportion of securities in a portfolio depends not only on their means and variances, but on the interrelationships or covariance. Thus, covariances between securities as well as returns and variances are calculated as input in portfolio optimization. Markowitz portfolio theory uses an equally weighted scheme for calculating the parameters listed above. Once the input parameters are obtained, both the risk and the return on any portfolio consisting of security combinations are calculated as follows, where μ_p is the return, $\sigma_p{}^2$ is the variance on the portfolio and ρ_{ij} symbolizes the correlation coefficient between the assets i and j.

$$\mu_p = \sum_{i=1}^{n} \mu_i x_i \tag{1}$$

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_i \sigma_j x_i x_j \rho_{ij} \tag{2}$$

The goal of portfolio optimization is to find a combination of assets (x_i: portfolio weights of each asset) that minimizes the standard deviation of the portfolio return for any given level of expected return or, in other words, a combination of assets that maximizes the expected return of the portfolio for any given level of risk.

The optimization problem usually faces certain constraints, for example, a budget constraint or a no short-selling constraint. Budget constraint requires that the weights of each security in a portfolio sum up to 1 and a no short-selling constraint requires the weight of each security in a portfolio to be non-negative. Considering the objective of minimizing the variance of the portfolio for a given level of expected return (μ_0) within the budget and no short-selling constraints, the portfolio selection problem can be summarized as follows.

$$\begin{aligned}
\text{Min} \quad & \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_i \sigma_j \rho_{ij} \\
& \mu_p \geq \mu_0 \\
& \sum_{i=1}^{n} x_i = 1 \\
& x_i \geq 0 \qquad ;\forall\, i = 1,2,\cdots,n
\end{aligned} \tag{3}$$

This is a quadratic programming problem and once the model is set up, it can be solved easily using the EXCEL add-in Solver or any other optimization tool such as LINGO or MATLAB. Changing the level of expected return (μ0) and solving the model iteratively the efficient frontier can be obtained.

3. Exponentially Weighted Scheme

Assigning equal weight to each datum prevents modelling unusual effects. The EWMA technique is used for measuring volatility by weighting recent observations more heavily than the distant ones. Mandelbrot (1963) and Fama (1965) encountered significant findings about financial asset prices. Considerably high price changes are followed by high price changes and conversely, low price changes are followed by low price changes because of the serial correlation between financial asset returns. The exponentially weighted moving average (EWMA) model assumes that volatility is not constant during the investment horizon.

Therefore, the EWMA technique provides a more accurate volatility model for the relation between asset returns. Returns of recent observations to distant ones are weighted by multiplying each term by $\lambda^0, \lambda^1, \lambda^2, \lambda^3, \ldots, \lambda^j, \ldots (0 < \lambda < 1)$ respectively and the product is divided by the term given below.

$$\sum_{j=1}^{\infty} \lambda^{j-1} \cong \frac{1}{1-\lambda} \tag{4}$$

Thus, the exponentially weighted scheme, where $\sigma_{1,t+1}$ is the standard deviation of a naive series at time t+1 and r1,t is the return of the series at time t, can be written as follows.

$$\sigma_{1,t+1} = (1-\lambda)\left((r_{1,t}-\bar{r}_1)^2 + \lambda(r_{1,t-1}-\bar{r}_1)^2 + \lambda^2(r_{1,t-2}-\bar{r}_1)^2 + \lambda^3(r_{1,t-3}-\bar{r}_1)^2 + \ldots\right) \quad (5)$$

In addition to this, it is assumed that mean value of daily returns is equal to zero in financial markets (Jorion, 2000:101) $\left(E(r_{i,t+1}^2) = \sigma_{i,t}^2\right)$. The standard deviation of the series at time t+1 is calculated as follows.

$$\sigma_{1,t+1} = (1-\lambda)\sum_{i=0}^{\infty} \lambda^i r_{t-i}^2 \quad (6)$$

Thus, J.P. Morgan RiskMetrics calculates exponentially weighted volatility estimators within the equation given below for T observations.

$$\sigma_{1,t+1} = \sqrt{(1-\lambda)\sum_{t=1}^{T} \lambda^{t-1}(r_{1,t}-\bar{r}_1)^2} \quad (7)$$

Equation (7) emphasizes that within every new observation, the volatility term changes. In Engle's (1982) seminal work on the ARCH (Auto-Regressive Conditional Heteroscedasticity) model, the most recent observation gives information about the one-period forecast variance. The EWMA model is a particular type of GARCH (1,1) (Generalized ARCH) model proposed by Bollerslev (1986). The EWMA model is obtained assuming one of the parameters of GARCH (1,1) model equal to zero. Therefore, it is a simpler form of GARCH model since just one parameter (λ) is used as an input. RiskMetrics uses two different series for estimating and forecasting covariances and correlations. Exponentially weighted covariance estimators are computed as follows.

$$Cov(r_1, r_2) = (1-\lambda)\sum_{j=1}^{T} \lambda^{j-1}(r_{1,t}-\bar{r}_1)(r_{2,t}-\bar{r}_2) \quad (8)$$

This model assumes a linear relationship between two assets. RiskMetrics uses an exponentially weighted measure of correlation (Best, 1998:79). In contrast with the model proposed by RiskMetrics, Jorion (2000) calculates the variance estimator as follows by assuming the mean value of daily returns equal to zero.

$$Cov(r_1, r_2) = (1-\lambda)\sum_{j=1}^{T} \lambda^{j-1} r_{1,t} r_{2,t} \quad (9)$$

This model depends on the λ ($0<\lambda<1$) parameter, or decay factor, which directly affects the calculation of volatility. By setting the value for λ close to 1, the effects of recent observations become stronger compared to the distant ones.

RiskMetrics uses the decay factors, 0.94 for the daily data set and 0.97 for the monthly data set, and claims to provide superior forecasting accuracy. A higher decay factor provides more stable forecasts. These values have been chosen by minimizing the mean squared error over smoothed series (Penza and Bansal, 2001:133).

In addition to this, consideration of the time horizon is also important. This affects volatility, but since distant observations are multiplied by the increasing power of the decay factor, observations distant from a specified value become negligible. Basically, as n increases λ becomes negligible. Thus, the number of effective days (T) can be computed by using the following formula (*RiskMetrics*, 1996:93), where $1-\alpha$ is the confidence level. Since daily data is used in this study, particularly 0.94 is chosen as the decay factor. Further, portfolio risk is computed for alternative decay factor values and results are gathered in Table 7.

$$T = \frac{\ln \alpha}{\ln \lambda} \quad (10)$$

4. Generalised Autoregressive Conditional Heteroscedasticity Models (GARCH)

The exponential smoothing technique has several advantages over the slightly more complex models such as ARMA and ARCH. It is simple to use and update when new data becomes available. However, it has several disadvantages: specifically, forecasts from an exponentially weighted model do not converge on the long-term mean of the variable as the horizon increases and, generally, it is overly simplistic and inflexible.

The classical linear regression model assumes homoscedasticity, that is, the variance of errors has to be constant. It is unlikely in the context of financial time series that the variance of errors will be constant over time. A model that does not assume variance to be constant over time is the Auto-Regressive Conditional Heteroscedasticity (ARCH) model. A full model of ARCH(q) is given as follows.

$$y_t = \beta_1 + \sum_{j=2}^{p} \beta_j x_{jt} + u_t \qquad u_t \sim N(0, \sigma_t^2)$$
$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{q} \alpha_j u_{t-j}^2 \tag{11}$$

The ARCH model provides a framework for modelling and forecasting volatility in financial data. The first equation of the given model mimics a classical linear regression model. Residuals are obtained from this equation. The ARCH model is constructed by squaring the residuals and regressing them on q own lags.

Despite the superiority of the ARCH model over the EWMA technique in allowing the variance not to be constant over time, the ARCH model has rarely been used over the last decade due to a number of limitations. There is no clearly best approach on deciding the number of lags of the squared residuals (q) in the model. Additionally, the estimate of the conditional variance has to be strictly positive. The more parameters there are in the conditional variance equation, the more likely it is that some of them will have negative values. An extension of the ARCH(q) model which overcomes some of these problems is the Generalised Auto-Regressive Conditional Heteroscedasticity (GARCH) model developed by Bollersev (1986) and Taylor (1986). In contrast with ARCH, the GARCH model is extremely widely employed in the finance industry.

The GARCH model allows the conditional variance to be dependent upon previous own lags as well as the squared residuals. Consequently, the model is less likely to breach non-negativity constraints. GARCH(p,q) formulation can be summarized as follows.

$$y_t = \beta_1 + \sum_{j=2}^{p} \beta_j x_{jt} + u_t \qquad u_t \sim N(0, \sigma_t^2)$$
$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{q} \alpha_i u_{t-i}^2 + \sum_{j=1}^{p} \beta_j \sigma_{t-j}^2 \tag{12}$$

Generally, GARCH(1,1) model is sufficient to capture the volatility clustering in the data. The variances of each security can be estimated by using the GARCH(p,q) model but covariances cannot be estimated since they require multivariate techniques. To investigate how covariances move over time, the BEKK model proposed by Engle and Kroner (1995) is used.

5. Empirical Results

Equally weighted and exponentially weighted schemes were applied to 100 daily datapoints for fifteen securities from the Istanbul Stock Exchange XU030 index within the period 09.08.2005–30.12.2005. The list of securities taken into consideration with their risk and return characteristics is given in Table 1. There was no particular reason for selecting the period and the specified securities. The main idea was to compare different techniques with the same input data. However, the number of securities is limited in order to maintain the normality assumption since the normality assumption is the most important requirement in modern portfolio theory.

For daily data n=100 is sufficient according to the RiskMetrics methodology on choosing the decay factor (RiskMetrics, 1996:100). As well as the EWMA technique, the GARCH model was used to estimate the variance of each security and the covariance term between securities. A time varying covariance matrix was calculated and input into the Markowitz portfolio selection model.

In order to compare the results obtained from each model, the Markowitz approach was applied to the data. Equally weighted mean, variances and covariances between securities were computed as well. The covariance matrix for the equally weighted and exponentially weighted schemes and the GARCH(1,1) model is given in Table 3. The covariance matrix calculated was used as an input for the Markowitz portfolio selection model given by equation (3). The effect of inflation over the period was ignored.

Focusing on Table 1, the Return/Risk parameters for the securities GARAN, HURGZ, ISCTR and TSKB imply that the risk premium for these securities is higher. This result is supported by the Sharpe ratio given in Table 1. Using the Markowitz approach with equally weighted data for an expected return level of 0.004%, resulted in a portfolio consisting of 99.9% of these securities. By increasing the level of the expected return, the allocation to TSKB increased due to its higher return, however, increasing the level of expected return resulted in the portfolio primarily consisting of

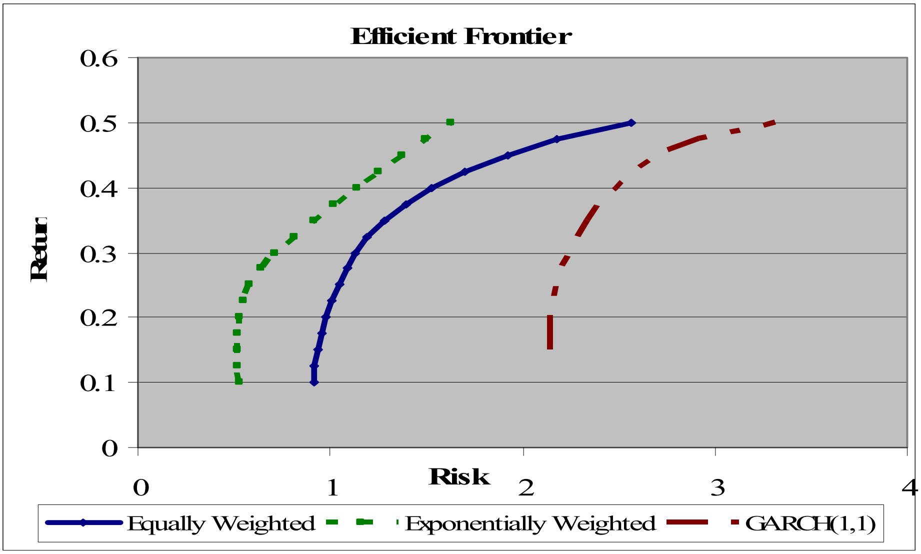
ULKER, DYHOL and TNSAS securities due to the lower risk/return ratios. The optimal portfolio depends on the investors' risk preferences; therefore, it is essential to obtain the efficient frontier. Efficient frontier is obtained by solving the portfolio selection problem given by equation (3) with respect to different expected return levels (μ0). The EXCEL add-in solver was used to solve the quadratic programming model and the optimal portfolios for different level of expected returns with their return and risk parameters are listed in Table 4.

In order to make the comparison between approaches on calculating the volatility, the quadratic model given by equation (3) was solved using exponentially weighted and GARCH(1,1) covariance matrices. Since the exponentially weighted scheme gives greater weight to the most recent observation, the performance of each security within closer investment horizons is taken into consideration. Risk and return characteristics of securities within different time horizons are given in Table 2 which shows that return/risk parameters for each security change slightly depending on the last 25 days' performance.

GARAN, HURGZ, ISCTR and TSKB securities still have the highest return/risk ratios whereas, AKBNK, TNSAS, SAHOL and TOASO securities take higher values compared to the equally weighted scheme. Thus, the effect of the exponentially weighted scheme on portfolio choices needs to be investigated. Optimal portfolios obtained by the exponentially weighted covariance matrix for different levels of expected returns with their return and risk parameters are listed in Table 5. Different asset allocations result from the changes on the return and risk parameters within 25 days' performance. Using the Markowitz approach with the exponentially weighted scheme for an expected return level of 0.004%, resulted in a portfolio consisting of 91.5% of GARAN, TNSAS and TSKB securities. In contrast to the equally weighted scheme, TNSAS and TOASO became part of the portfolio and risk parameters of optimal portfolios took lower values.

Finally, the quadratic model given by equation (3) was solved by using the covariance matrix calculated by GARCH(1,1). Optimal portfolios obtained by the GARCH(1,1) covariance matrix for different levels of expected returns with their return and risk parameters are listed in Table 6. Efficient frontier is obtained by solving the portfolio selection problem given by equation (3) with respect to different expected return levels (μ0) with exponentially weighted and GARCH(1,1) covariance matrices. Figure 1 illustrates the combined results obtained from equally and exponentially weighted schemes.

Figure 1: Efficient frontier for equally and exponentially weighted and GARCH(1,1) schemes



The solid line illustrates the efficient frontier obtained by the equally weighted scheme whereas the dashed line shows the efficient frontier obtained by the exponentially weighted scheme. The remaining line shows the efficient frontier for the GARCH(1,1) covariance matrix.

Superimposing all three efficient frontiers together on the same (σ,μ) plane, makes it clear that the risk parameter of portfolios obtained by the exponentially weighted scheme is always lower than the ones obtained from the other schemes for every level of expected return. Thus, at the same level of expected return, one can always have less risky portfolios by using exponentially weighted covariance matrices.

In order to make the comparison more concrete, expected returns from each model were computed by setting $\sigma = 0.003$ and calculating the expected return for each model. Expected daily returns of 0.51083%, 0.94251% and 0.17485% for equally weighted, exponentially weighted and GARCH(1,1) schemes respectively were achieved. According to the findings, the exponentially weighted scheme is superior as a higher level of expected return is obtained. The findings of this paper are consistent with those obtained by Tse (1991).

Finally, different values of decay factor were used and the portfolio risk for each decay factor was calculated. Results are summarized in Table 7.

6. Conclusion

In this paper, the use of exponentially weighted moving averages and the generalised auto-regressive conditional heteroscedasticity technique in portfolio selection was applied to a selection of stocks from the Istanbul Stock Exchange market and the performance of each model was compared. In order to make a comparison, optimal portfolios at each expected return level were obtained by classical Markowitz theory and findings were compared with two other volatility modelling techniques, EWMA and GARCH.

The results obtained in Tables 4 to 6 and illustrated in Figure 1 assert that at the same level of expected return, one can always have less risky portfolios by using exponentially weighted covariance matrices. Working with exponentially weighted data is superior to the equally weighted and GARCH(1,1) data since recent performance of securities is given greater weight in forecasting future performance and current market conditions are modelled more accurately. GARCH forecasts seem to react slower to the changes in volatility. Markowitz (1952:91) pointed to a way for calculating reasonable μ and σ parameters and the exponentially weighted scheme provides a more efficient way to calculate these parameters and respond to stock market changes.

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Table 1: Risk and return characteristics of securities taken into consideration

Securities	**Arithmetic Mean %**	**Geometric Mean %**	**Standard Deviation %**	**Return / Risk**	**Alfa(α)**	**Beta(β)**	**Sharpe Ratio %**
AKBNK	0.3380	0.3736	2.6816	0.1260	-0.0769	1.4351	0.1260
ARCLK	0.1428	0.1693	2.3264	0.0614	-0.0958	0.8252	0.0614
DYHOL	0.3749	0.4127	2.7682	0.1354	0.0857	1.0004	0.1354
GARAN	0.3370	0.3473	1.4360	0.2347	0.1825	0.5344	0.2347
HURGZ	0.4273	0.4495	2.1170	0.2019	0.2081	0.7582	0.2019
ISCTR	0.4457	0.4823	2.7133	0.1643	0.0182	1.4785	0.1643
ISGYO	0.2836	0.3083	2.2321	0.1270	0.0295	0.8788	0.1270
MIGRS	0.2026	0.2339	2.5161	0.0805	-0.0650	0.9255	0.0805
SAHOL	0.3214	0.3488	2.3469	0.1370	-0.0127	1.1558	0.1370
SISE	0.0660	0.0847	1.9339	0.0341	-0.1926	0.8945	0.0341
TNSAS	0.1922	0.2041	1.5865	0.1286	0.1314	0.2100	0.1211
TOASO	0.3023	0.3261	2.1927	0.1487	0.0401	0.9071	0.1379
TSKB	0.5121	0.5512	2.8222	0.1953	0.1594	1.2200	0.1815
ULKER	0.1843	0.2035	1.9770	0.1029	0.0164	0.5806	0.0932
VESTL	-0.0352	-0.0294	1.0797	-0.0272	-0.1576	0.4234	-0.0326

Table 2: Risk and return characteristics of securities within different time horizons

Time Horizon	Parameters	AKBNK	ARCLK	DYHOL	GARAN	HURGZ	ISCTR	ISGYO	MIGRS	SAHOL	SISE	TNSAS	TOASO	TSKB	ULKER	VESTL
Last 25 Days	Return	0.0043	0.0037	0.0104	0.0045	0.0037	0.0047	0.0037	0.0011	0.0005	0.0028	0.0000	-0.0003	0.0025	0.0022	0.0014
	Risk	0.0229	0.0201	0.0275	0.0135	0.0212	0.0272	0.0151	0.0258	0.0184	0.0148	0.0048	0.0153	0.0115	0.0133	0.0081
	Return/Risk	0.1872	0.1856	0.3779	0.3331	0.1729	0.1738	0.2437	0.0445	0.0264	0.1882	0.0000	-0.0170	0.2163	0.1687	0.1663
Last 50 Days	Return	0.0056	0.0020	0.0092	0.0040	0.0056	0.0057	0.0035	0.0018	0.0037	0.0032	0.0005	0.0017	0.0039	0.0010	0.0011
	Risk	0.0222	0.0187	0.0299	0.0136	0.0243	0.0253	0.0186	0.0228	0.0207	0.0177	0.0056	0.0203	0.0252	0.0131	0.0090
	Return/Risk	0.2548	0.1093	0.3066	0.2951	0.2303	0.2249	0.1872	0.0809	0.1803	0.1811	0.0908	0.0838	0.1536	0.0761	0.1229
Last 100 Days	Return	0.0034	0.0014	0.0037	0.0034	0.0043	0.0044	0.0028	0.0020	0.0032	0.0007	0.0019	0.0030	0.0051	0.0018	-0.0004
	Risk	0.0268	0.0230	0.0275	0.0144	0.0211	0.0271	0.0223	0.0251	0.0235	0.0194	0.0154	0.0219	0.0280	0.0196	0.0109
	Return/Risk	0.1260	0.0620	0.1359	0.2344	0.2020	0.1638	0.1269	0.0808	0.1367	0.0340	0.1249	0.1380	0.1828	0.0939	-0.0325

Table 3: Covariance matrices for equally weighted, exponentially weighted and GARCH(1,1) schemes respectively($*10^4$)

	AKBNK	ARCLK	DYHOL	GARAN	HURGZ	ISCTR	ISGYO	MIGRS	SAHOL	SISE	TNSAS	TOASO	TSKB	ULKER	VESTL
AKBNK	7.1002	2.5959	3.8626	1.5554	2.3576	5.7064	3.2438	3.0571	4.3525	3.4492	0.4669	3.2297	4.2021	1.9989	1.4049
ARCLK	2.5959	5.2418	1.8940	1.0907	1.2382	2.3426	1.8203	2.9692	2.3628	2.0801	0.7524	2.4107	2.4138	1.2812	1.3025
DYHOL	3.8626	1.8940	7.5099	1.1347	2.3022	3.5646	2.2621	2.3574	2.3315	2.4106	0.3188	2.2336	3.9205	1.4351	1.5531
GARAN	1.5554	1.0907	1.1347	2.0403	1.1735	1.7726	1.2047	1.2466	1.5673	0.9101	0.7772	0.9171	1.9413	0.7064	0.6002
HURGZ	2.3576	1.2382	2.3022	1.1735	4.4105	2.9775	2.0507	2.4582	2.3398	1.7862	0.2265	1.9141	2.8286	1.1294	0.7786
ISCTR	5.7064	2.3426	3.5646	1.7726	2.9775	7.2930	3.5531	2.9653	4.2498	3.3308	0.6091	3.3114	4.4248	2.3200	1.4508
ISGYO	3.2438	1.8203	2.2621	1.2047	2.0507	3.5531	4.9311	2.6069	2.8866	2.0808	0.5929	2.2721	2.8798	1.5618	0.8753
MIGRS	3.0571	2.9692	2.3574	1.2466	2.4582	2.9653	2.6069	6.2177	2.2854	2.4895	1.5438	2.5724	2.8269	1.1795	1.0092
SAHOL	4.3525	2.3628	2.3315	1.5673	2.3398	4.2498	2.8866	2.2854	5.4523	2.4473	0.1594	2.9843	3.3838	1.4546	1.1502
SISE	3.4492	2.0801	2.4106	0.9101	1.7862	3.3308	2.0808	2.4895	2.4473	3.7366	0.6591	2.0604	2.8329	1.3840	0.9466
TNSAS	0.4669	0.7524	0.3188	0.7772	0.2265	0.6091	0.5929	1.5438	0.1594	0.6591	2.3387	0.6755	0.7548	0.4855	0.2236
TOASO	3.2297	2.4107	2.2336	0.9171	1.9141	3.3114	2.2721	2.5724	2.9843	2.0604	0.6755	4.7365	2.5321	1.5438	0.9359
TSKB	4.2021	2.4138	3.9205	1.9413	2.8286	4.4248	2.8798	2.8269	3.3838	2.8329	0.7548	2.5321	7.7346	2.4636	1.9693
ULKER	1.9989	1.2812	1.4351	0.7064	1.1294	2.3200	1.5618	1.1795	1.4546	1.3840	0.4855	1.5438	2.4636	3.8052	0.7180
VESTL	1.4049	1.3025	1.5531	0.6002	0.7786	1.4508	0.8753	1.0092	1.1502	0.9466	0.2236	0.9359	1.9693	0.7180	1.1665
AKBNK	4.0815	1.1731	2.4481	1.0771	1.6138	3.0765	1.4763	1.2986	2.3948	1.5579	0.2274	1.6540	1.2805	0.8893	0.3646
ARCLK	1.1731	3.6074	0.8702	0.8903	0.7121	1.4782	0.9222	1.8913	1.1602	1.4503	0.5945	1.2583	0.6006	1.0438	0.5102
DYHOL	2.4481	0.8702	8.2597	0.7662	1.9061	2.2191	1.4700	1.4153	1.0594	1.1610	0.2006	1.4341	1.6592	0.4652	0.9560
GARAN	1.0771	0.8903	0.7662	1.7030	1.3648	1.7038	1.0830	0.8921	1.3321	0.5204	0.2274	0.4325	1.0814	0.7722	0.3509
HURGZ	1.6138	0.7121	1.9061	1.3648	4.0951	2.8649	1.9325	2.2681	1.6224	1.0008	0.1282	1.2400	1.7740	1.1586	0.5052
ISCTR	3.0765	1.4782	2.2191	1.7038	2.8649	6.5982	2.7031	2.8222	2.6794	1.6082	0.5110	1.9537	1.7015	1.4056	0.6954
ISGYO	1.4763	0.9222	1.4700	1.0830	1.9325	2.7031	2.9194	1.6723	1.5658	1.1821	0.2140	1.0638	1.1611	0.7395	0.4633
MIGRS	1.2986	1.8913	1.4153	0.8921	2.2681	2.8222	1.6723	5.0567	1.5250	1.3287	0.6198	1.8438	0.6294	0.9274	0.1903
SAHOL	2.3948	1.1602	1.0594	1.3321	1.6224	2.6794	1.5658	1.5250	2.8569	1.1336	0.2108	1.5048	1.5186	0.8420	0.3692
SISE	1.5579	1.4503	1.1610	0.5204	1.0008	1.6082	1.1821	1.3287	1.1336	2.0277	0.2782	0.9780	0.6254	0.6125	0.0929
TNSAS	0.2274	0.5945	0.2006	0.2274	0.1282	0.5110	0.2140	0.6198	0.2108	0.2782	0.3106	0.3920	0.1197	0.3007	0.1278
TOASO	1.6540	1.2583	1.4341	0.4325	1.2400	1.9537	1.0638	1.8438	1.5048	0.9780	0.3920	3.0927	0.8642	0.8975	0.1996

TSKB	1.2805	0.6006	1.6592	1.0814	1.7740	1.7015	1.1611	0.6294	1.5186	0.6254	0.1197	0.8642	2.9329	0.7582	0.7543
ULKER	0.8893	1.0438	0.4652	0.7722	1.1586	1.4056	0.7395	0.9274	0.8420	0.6125	0.3007	0.8975	0.7582	1.7030	0.3057
VESTL	0.3646	0.5102	0.9560	0.3509	0.5052	0.6954	0.4633	0.1903	0.3692	0.0929	0.1278	0.1996	0.7543	0.3057	0.6790
AKBNK	10.0183	5.2480	6.9925	9.2009	6.6158	7.0124	6.0163	4.2332	6.8491	6.6605	5.0553	5.0043	6.8184	6.8310	4.2924
ARCLK	5.2480	16.9771	4.7042	5.1365	4.7423	5.7147	4.2703	3.7650	5.7695	4.6307	4.1819	4.3107	4.6550	4.7821	3.1870
DYHOL	6.9925	4.7042	14.8183	7.2350	6.1482	6.3166	6.7655	4.0411	5.8874	5.3875	4.1481	4.4248	4.1481	6.4205	3.4725
GARAN	9.2009	5.1365	7.2350	15.2411	7.0218	7.6841	6.7391	4.6785	7.0544	6.8445	4.9192	5.1984	7.6874	7.1562	4.5808
HURGZ	6.6158	4.7423	6.1482	7.0218	8.1127	6.4407	5.2890	4.2241	5.5216	6.2542	4.2677	4.9083	5.8974	5.4228	3.8499
ISCTR	7.0124	5.7147	6.3166	7.6841	6.4407	12.3407	6.0711	4.4999	6.7859	5.9656	5.2034	5.4950	6.8525	6.5381	4.6360
ISGYO	6.0163	4.2703	6.7655	6.7391	5.2890	6.0711	10.3165	4.0717	5.4805	6.2589	4.1442	4.8889	5.3541	5.3948	4.3816
MIGRS	4.2332	3.7650	4.0411	4.6785	4.2241	4.4999	4.0717	7.1527	4.1650	4.4989	3.8412	3.3499	4.4343	3.6516	3.0445
SAHOL	6.8491	5.7695	5.8874	7.0544	5.5216	6.7859	5.4805	4.1650	10.3600	5.8178	5.1395	5.3716	6.2552	6.1692	4.2907
SISE	6.6605	4.6307	5.3875	6.8445	6.2542	5.9656	6.2589	4.4989	5.8178	9.1087	4.5268	5.0291	6.5689	6.1661	3.6609
TNSAS	5.0553	4.1819	4.1481	4.9192	4.2677	5.2034	4.1442	3.8412	5.1395	4.5268	10.6529	3.6345	5.0832	4.3135	3.1429
TOASO	5.0043	4.3107	4.4248	5.1984	4.9083	5.4950	4.8889	3.3499	5.3716	5.0291	3.6345	8.9047	4.6362	4.6897	4.1156
TSKB	6.8184	4.6550	4.1481	7.6874	5.8974	6.8525	5.3541	4.4343	6.2552	6.5689	5.0832	4.6362	12.8638	6.8057	4.5459
ULKER	6.8310	4.7821	6.4205	7.1562	5.4228	6.5381	5.3948	3.6516	6.1692	6.1661	4.3135	4.6897	6.8057	14.0689	4.0118
VESTL	4.2924	3.1870	3.4725	4.5808	3.8499	4.6360	4.3816	3.0445	4.2907	3.6609	3.1429	4.1156	4.5459	4.0118	8.4519

Table 4: Optimal portfolios for equally weighted scheme

	Return%	Risk %	AKBNK	ARCLK	DYHOL	GARAN	HURGZ	ISCTR	ISGYO	MIGRS	SAHOL	SISE	TNSAS	TOASO	TSKB	ULKER	VESTL
1	0.1000	0.9112	0.000	0.000	0.000	0.125	0.043	0.000	0.000	0.000	0.000	0.000	0.252	0.000	0.000	0.053	0.527
2	0.1250	0.9180	0.000	0.000	0.161	0.064	0.000	0.000	0.000	0.000	0.000	0.255	0.000	0.000	0.058	0.462	0.000
3	0.1500	0.9316	0.000	0.000	0.000	0.198	0.084	0.000	0.000	0.000	0.000	0.000	0.257	0.000	0.000	0.065	0.396
4	0.1750	0.9520	0.000	0.000	0.000	0.234	0.104	0.000	0.000	0.000	0.000	0.000	0.259	0.003	0.000	0.069	0.330
5	0.2000	0.9786	0.000	0.000	0.000	0.269	0.120	0.000	0.000	0.000	0.000	0.000	0.260	0.013	0.000	0.073	0.264
6	0.2250	1.0108	0.000	0.000	0.003	0.303	0.135	0.000	0.000	0.000	0.000	0.000	0.262	0.022	0.000	0.076	0.198
7	0.2500	1.0475	0.000	0.000	0.015	0.333	0.148	0.000	0.000	0.000	0.000	0.000	0.264	0.029	0.000	0.079	0.132
8	0.2750	1.0881	0.000	0.000	0.027	0.364	0.159	0.000	0.000	0.000	0.000	0.000	0.267	0.035	0.000	0.081	0.066
9	0.3000	1.1319	0.000	0.000	0.039	0.394	0.171	0.000	0.000	0.000	0.000	0.000	0.270	0.042	0.000	0.084	0.000
10	0.3250	1.1937	0.000	0.000	0.046	0.449	0.219	0.000	0.000	0.000	0.000	0.000	0.207	0.042	0.013	0.024	0.000
11	0.3500	1.2797	0.000	0.000	0.038	0.476	0.253	0.000	0.000	0.000	0.000	0.000	0.142	0.029	0.063	0.000	0.000
12	0.3750	1.3905	0.000	0.000	0.027	0.502	0.289	0.000	0.000	0.000	0.000	0.000	0.059	0.007	0.116	0.000	0.000
13	0.4000	1.5240	0.000	0.000	0.002	0.471	0.328	0.013	0.000	0.000	0.000	0.000	0.000	0.000	0.187	0.000	0.000
14	0.4250	1.7021	0.000	0.000	0.000	0.303	0.370	0.028	0.000	0.000	0.000	0.000	0.000	0.000	0.300	0.000	0.000
15	0.4500	1.9221	0.000	0.000	0.000	0.133	0.412	0.043	0.000	0.000	0.000	0.000	0.000	0.000	0.412	0.000	0.000
16	0.4750	2.1756	0.000	0.000	0.000	0.000	0.399	0.033	0.000	0.000	0.000	0.000	0.000	0.000	0.568	0.000	0.000
17	0.5000	2.5657	0.000	0.000	0.000	0.000	0.128	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.872	0.000	0.000

Table 5: Optimal portfolios for exponentially weighted scheme

	Return%	Risk %	AKBNK	ARCLK	DYHOL	GARAN	HURGZ	ISCTR	ISGYO	MIGRS	SAHOL	SISE	TNSAS	TOASO	TSKB	ULKER	VESTL
1	0.1000	0.5275	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.048	0.574	0.000	0.000	0.000	0.378
2	0.1250	0.5151	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.026	0.694	0.000	0.000	0.000	0.280
3	0.1500	0.5152	0.000	0.000	0.000	0.002	0.018	0.000	0.000	0.000	0.001	0.003	0.767	0.000	0.003	0.000	0.207
4	0.1750	0.5214	0.000	0.000	0.000	0.005	0.020	0.000	0.000	0.000	0.000	0.000	0.803	0.000	0.030	0.000	0.141
5	0.2000	0.5326	0.000	0.000	0.000	0.008	0.022	0.000	0.000	0.000	0.000	0.000	0.838	0.000	0.058	0.000	0.074
6	0.2250	0.5486	0.000	0.000	0.000	0.011	0.024	0.000	0.000	0.000	0.000	0.000	0.872	0.000	0.085	0.000	0.007
7	0.2500	0.5798	0.000	0.000	0.000	0.027	0.029	0.000	0.000	0.000	0.000	0.000	0.795	0.000	0.148	0.000	0.000
8	0.2750	0.6376	0.000	0.000	0.000	0.044	0.035	0.000	0.000	0.000	0.000	0.000	0.706	0.000	0.215	0.000	0.000
9	0.3000	0.7162	0.000	0.000	0.000	0.061	0.040	0.000	0.000	0.000	0.000	0.000	0.617	0.000	0.281	0.000	0.000
10	0.3250	0.8094	0.000	0.000	0.000	0.078	0.046	0.000	0.000	0.000	0.000	0.000	0.528	0.000	0.348	0.000	0.000
11	0.3500	0.9127	0.000	0.000	0.000	0.096	0.050	0.000	0.000	0.000	0.000	0.000	0.436	0.004	0.414	0.000	0.000
12	0.3750	1.0228	0.000	0.000	0.000	0.114	0.053	0.001	0.000	0.000	0.000	0.000	0.340	0.015	0.477	0.000	0.000
13	0.4000	1.1380	0.000	0.000	0.000	0.128	0.051	0.012	0.000	0.000	0.000	0.000	0.247	0.022	0.540	0.000	0.000
14	0.4250	1.2554	0.000	0.000	0.000	0.142	0.049	0.023	0.000	0.000	0.000	0.000	0.155	0.030	0.602	0.000	0.000
15	0.4500	1.3756	0.000	0.000	0.000	0.155	0.047	0.034	0.000	0.000	0.000	0.000	0.062	0.037	0.665	0.000	0.000
16	0.4750	1.4982	0.000	0.000	0.000	0.134	0.049	0.051	0.000	0.000	0.000	0.000	0.000	0.024	0.742	0.000	0.000
17	0.5000	1.6316	0.000	0.000	0.000	0.006	0.053	0.079	0.000	0.000	0.000	0.000	0.000	0.000	0.861	0.000	0.000

Table 6: Optimal portfolios for GARCH(1,1) scheme

	Return%	Risk %	AKBNK	ARCLK	DYHOL	GARAN	HURGZ	ISCTR	ISGYO	MIGRS	SAHOL	SISE	TNSAS	TOASO	TSKB	ULKER	VESTL
1	0.1500	2.1450	0.000	0.054	0.031	0.000	0.055	0.000	0.020	0.300	0.000	0.000	0.129	0.142	0.000	0.034	0.235
2	0.1750	2.1451	0.000	0.054	0.031	0.000	0.055	0.000	0.020	0.300	0.000	0.000	0.129	0.142	0.000	0.034	0.235
3	0.2000	2.1501	0.000	0.047	0.037	0.000	0.096	0.000	0.024	0.292	0.000	0.000	0.124	0.154	0.010	0.024	0.193
4	0.2250	2.1628	0.000	0.040	0.044	0.000	0.122	0.000	0.026	0.282	0.000	0.000	0.117	0.163	0.035	0.012	0.158
5	0.2500	2.1828	0.000	0.034	0.051	0.000	0.148	0.000	0.027	0.273	0.000	0.000	0.110	0.173	0.060	0.000	0.123
6	0.2750	2.2099	0.000	0.027	0.056	0.000	0.176	0.000	0.028	0.262	0.000	0.000	0.102	0.181	0.084	0.000	0.084
7	0.3000	2.2445	0.000	0.019	0.062	0.000	0.203	0.000	0.029	0.251	0.000	0.000	0.094	0.189	0.108	0.000	0.045
8	0.3250	2.2859	0.000	0.012	0.067	0.000	0.228	0.003	0.029	0.241	0.000	0.000	0.086	0.197	0.131	0.000	0.006
9	0.3500	2.3365	0.000	0.000	0.074	0.000	0.266	0.023	0.013	0.207	0.000	0.000	0.063	0.190	0.164	0.000	0.000
10	0.3750	2.4018	0.000	0.000	0.082	0.000	0.306	0.044	0.000	0.158	0.000	0.000	0.032	0.176	0.202	0.000	0.000
11	0.4000	2.4819	0.000	0.000	0.087	0.000	0.347	0.063	0.000	0.105	0.000	0.000	0.000	0.159	0.240	0.000	0.000
12	0.4250	2.5780	0.000	0.000	0.089	0.000	0.392	0.082	0.000	0.025	0.000	0.000	0.000	0.131	0.280	0.000	0.000
13	0.4500	2.6992	0.000	0.000	0.077	0.000	0.437	0.111	0.000	0.000	0.000	0.000	0.000	0.029	0.346	0.000	0.000
14	0.4750	2.9163	0.000	0.000	0.000	0.000	0.331	0.120	0.000	0.000	0.000	0.000	0.000	0.000	0.550	0.000	0.000
15	0.5000	3.3420	0.000	0.000	0.000	0.000	0.056	0.092	0.000	0.000	0.000	0.000	0.000	0.000	0.852	0.000	0.000

Table 7: Minimum risks for alternative decay factors at various expected returns

Return%: 0.1		Return%: 0.2		Return%: 0.3		Return%: 0.4		Return%: 0.5	
Lambda	Risk %	Lambda	Risk %	Lambda	Risk %	Lambda	Risk %	Lambda	Risk %
0.10	0.0192	0.10	0.0553	0.10	0.0969	0.10	0.3349	0.10	0.6747
0.15	0.0282	0.15	0.0710	0.15	0.1198	0.15	0.3936	0.15	0.7925
0.20	0.0392	0.20	0.0841	0.20	0.1410	0.20	0.4359	0.20	0.8765
0.25	0.0526	0.25	0.0974	0.25	0.1611	0.25	0.4675	0.25	0.9370
0.30	0.0689	0.30	0.1113	0.30	0.1828	0.30	0.4915	0.30	0.9796
0.35	0.0886	0.35	0.1299	0.35	0.2085	0.35	0.5105	0.35	1.0078
0.40	0.1128	0.40	0.1542	0.40	0.2407	0.40	0.5261	0.40	1.0238
0.45	0.1410	0.45	0.1855	0.45	0.2809	0.45	0.5404	0.45	1.0290
0.50	0.1745	0.50	0.2234	0.50	0.3336	0.50	0.5556	0.50	1.0248
0.55	0.2128	0.55	0.2619	0.55	0.3733	0.55	0.5735	0.55	1.0119
0.60	0.2563	0.60	0.2979	0.60	0.4123	0.60	0.5943	0.60	0.9914
0.65	0.3073	0.65	0.3315	0.65	0.4409	0.65	0.6180	0.65	0.9647
0.70	0.3604	0.70	0.3639	0.70	0.4593	0.70	0.6367	0.70	0.9350
0.75	0.4086	0.75	0.3969	0.75	0.4684	0.75	0.6418	0.75	0.9061
0.80	0.4433	0.80	0.4276	0.80	0.4713	0.80	0.6416	0.80	0.8904
0.85	0.4631	0.85	0.4450	0.85	0.4797	0.85	0.6674	0.85	0.9329
0.90	0.4821	0.90	0.4666	0.90	0.5450	0.90	0.8211	0.90	1.1666
0.94	0.5275	0.94	0.5326	0.94	0.7162	0.94	1.1375	0.94	1.6317
0.97	0.6566	0.97	0.7184	0.97	0.9229	0.97	1.4090	0.97	2.1080