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From Market Crashes to Heart Attacks: On the Empirics of Nonlinear Dynamics and Chaos in Nature and Society

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Abstract. The paper is aimed at highlighting some of the subtleties of empirical analysis in complex dynamics. It focuses, on the one hand, on the critical assumptions underlying the chaos tests used in applied research. On the other hand, three examples of time series data analysis from the fields of economics, finance and cardiology are provided in order to investigate the characteristics of the data generating processes involved and to illustrate the difficulties encountered in numerical analyses.

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1. Introduction

During the last three decades or so, researchers working in a variety of fields such as mechanics, chemistry, population biology, statistics, economics and medicine have developed models by means of which deeper insights into the working of some seemingly complex phenomena could be obtained. These systems appear to exhibit incongruous and complex dynamics, a behavior difficult to analyze with the traditional tools of differential (or difference) equations. The terms “chaos”, "strange attractors", "complex dynamics" have been used in the literature to characterize these processes that look random but in fact possess deterministic dynamics. In this context, chaos theory revealed that some dynamical systems with simple deterministic structures could display such complex behavior that would pass traditional tests of randomness.

While the extant literature on chaos and complex dynamics is too large and too diverse to be surveyed meaningfully within the confines of a journal article, it is possible to focus on some critical issues underlying applied research in order to highlight some of its limitations. Applied research often fails to mention these limitations explicitly and the intricate details of empirical analysis often hide the limitations of the statistical tests used, as well as the special assumptions made about the nature of the data generating processes.

The purpose of this paper is two-fold. First, it attempts to provide a brief introduction to the empirics of the nonlinear dynamics/chaos literature with an eye toward highlighting the difficulties encountered in applied research. Second, it provides three applications of chaos theory to economics/finance and cardiology so as to illustrate the limitations of numerical analysis. Section II provides an elementary introduction to chaos theory and nonlinear dynamic analysis. Section III offers an overview of some of the most popular tools of numerical analysis and chaos tests. As an example of the application of chaos theory to economic time series, the empirical results of exchange rate analysis are given in Section IV. Section V focuses on the stock market data based on the S&P 500 and the Dow Jones Industrial Averages Indices. Section VI considers the cardiology literature and presents the results of the correlation dimension estimates and the Brock-Dechert-Scheinkman-LeBaron (BDSL) statistics for two samples of heartbeat data. Some concluding remarks are given in Section VII.

2. Deterministic Randomness: Chaos and Nonlinear Dynamics

Chaos is a nonlinear deterministic process that looks random, a case in which a dynamic mechanism yields a time path so complicated that it passes most standard tests of randomness. Chaos theory shows that a simple relationship that is deterministic but nonlinear, such as a first-order nonlinear difference equation, can yield an extremely complex time path.

Chaotic time paths often have the following features:

i. A trajectory can sometimes display sharp qualitative changes in behavior like those associated with large random disturbances.
ii. A time path is extremely sensitive to microscopic changes in the values of the parameters, which is known as the sensitive dependence on initial conditions.
iii. A time path may never return to any point it had previously traversed, but displays an oscillatory pattern in a bounded region.

It is well known that linear deterministic models can only generate four types of time path: (i) oscillatory and stable (converging with oscillations of decreasing amplitude toward some fixed equilibrium value), (ii) oscillatory and explosive (cycles of ever-increasing amplitude), (iii) non oscillatory and stable, (iv) non-oscillatory and explosive. In reality however, time paths of several dynamic systems are much more complicated than these four cases. Research in chaotic dynamics has been partly motivated by the realization that dynamic modeling should go beyond linear differential - or difference in the case of discrete systems - equations in order to be able to adequately capture the underlying dynamics of seemingly more complex systems.

There exists a voluminous literature on chaos and nonlinear analysis and their applications have found fertile grounds in many fields. Apart from a plethora of articles published in many science and mathematics journals, Alligood et al. (1997), Devaney (1992), Hilborn (1994) and Ruelle (1989), Sprott (2003) provide a good introduction to complex dynamics. On the other hand, Brock (1986), Baumol and Benhabib (1989), Benhabib (1992), Brock and Malliaris (1989), Lorenz (1989), Medio (1992), Day (1994), are among the earlier authors in the chaos studies in economics. Similarly, many studies in cardiology have recently attempted to apply the findings of nonlinear dynamics and chaos to the analysis of heartbeat time series

(Lefebvre et al., 1993; Peng et al., 1995; Pincus, 1994 and Yamada et al., 2000).

Formally speaking, a chaotic system has a dense collection of points with periodic orbits, and is topologically transitive. The emergence of chaos in dynamic systems is captured by the following well-known theorem due to Li and Yorke (1975):

Theorem (Li-Yorke): Let a function θ such that $x_{t+1} = \theta(x_t)$ be continuous map on an interval $J \to J \subset R$. Assume that there exits a point $x \in J$ such that

$$\theta^3(x) \leq x < \theta(x) < \theta^2(x).$$

Then for every $k = 1, 2, 3, \ldots$, there exits a k-periodic trajectory in j and θ is chaotic on an uncountable scrambled set $S \subset J$.

Insofar as empirical analysis of time series is concerned, the importance of chaos theory and nonlinear dynamics has been to demonstrate that some deterministic processes can also have white noise properties. An example of such a process is the tent map, which is generated by

$$\begin{aligned} x_t &= a^{-1} x_{t-1}, && \text{if } 0 \leq x_{t-1} < a \\ &= (1-a)^{-1}(1-x_{t-1}), && \text{if } a \leq x_{t-1} \leq 1 \end{aligned} \tag{1}$$

Time series data generated by another well-known nonlinear map, the logistic map, exhibit similar properties. The logistic map is given by

$$x_t = w\, x_{t-1}\,(1 - x_{t-1}) \tag{2}$$

with some initial value x_0 in the range (0,1) and parameter value w in the range (0, 4).

For small values of w, the system is stable and well behaved. But as the value of w approaches 4, the system enters into a chaotic zone. These two maps are the most common textbook examples of chaotic dynamics, available from several introductory studies on chaos (Baumol and Benhabib, 1989; Lorenz, 1989).

Table 1 provides the estimated autocorrelations and partial autocorrelations for the tent and the logistic maps. The autocorrelations of the data generated by both of these maps are small and insignificantly different from zero, indicating that these series have white noise properties. However, x_t is clearly not an iid (independently and identically distributed - or random) process, as it is generated by a nonlinear deterministic process.

The simplest chaotic processes are the tent map and the logistic map given in (1) and (2) respectively. These maps generate low dimensional chaos where the nonlinear structure is easily detected. In addition to these univariate chaotic systems, there are also multivariate systems such as the Hénon and the Lorenz maps. There are, however, high dimensional chaotic systems that are harder to detect with finite data.

As already mentioned above, the most important characteristics of chaotic behavior include a fractal structure, sensitive dependence on initial conditions, and time dependent feedback systems. A fractal structure consists of two major features: self-similarity and lack of smoothness. Self-similarity refers to a system that always looks the same regardless of how many times the system is magnified. On the other hand, lack of smoothness relates to the disconnected appearance of fractals. Sensitivity dependences on the initial conditions correspond to accuracy in terms of a measurement of original state of conditions. In other words, a marginal error made in data collection or data itself could result in devastating wrong forecasting results. Time dependent feedback systems refer to the forecasting results that always depend upon what happened in the past.

3. Detecting Chaos: Some Popular Tools of Numerical Analysis

How can we test for low dimensional chaos? One way is to observe that chaotic maps fill much less space in higher dimension and have strange attractors. A strange attractor of a chaotic dynamical system is a compact set, S such that almost all initial conditions in the neighborhood of S converge to S.

Consider two sets of data: $\{y_t\}$ is generated by the tent map, and $\{z_t\}$ is uniformly and randomly distributed on the interval [0,1]. If we plot y_t in a one-dimensional space, it is uniform over [0,1] and so fills as much space as z_t does. Now, consider the pairs (y_{t-1}, y_t) and (z_{t-1}, z_t). When we plot them in a two-dimensional space, the data from the Tent map will fall on a tent-like graph, while the data from the uniform random variable will fall uniformly in the unit square $[0,1]\times[0,1]$. When the chaotic process is more complex, one needs to consider the data in higher dimensions.

a. Correlation Dimension Estimates

The above graphical exercise to distinguish between deterministic chaos and randomness is, however, not practical in higher dimensions. Grassberger and Procaccia (1983) therefore developed the notion of correlation dimension. Let $\{ x_t \}$ be a univariate time series. The correlation integral is defined as:

$$C(\varepsilon) = \lim_{N\to\infty} \frac{1}{N^2} \left\{ \text{number of pairs } (i, j) \text{ such that } | x_i - x_j | < \varepsilon \right\} \qquad (3)$$

Intuitively, $C(\varepsilon)$ measures the probability that any particular pair in the series is ε close. If for small values of ε, $C(\varepsilon)$ grows exponentially at the rate v

$$C(\varepsilon) \approx \varepsilon^{\upsilon} , \qquad (4)$$

where v is the correlation dimension.

A generalization is obtained forming m-histories of the series. An m-history is a point in m-dimensional space, and m is called the embedding dimension.

1-history: $x_t^1 = x_t$

2-history: $x_t^2 = (x_{t-1}, x_t)$

3-history: $x_t^3 = (x_{t-2}, x_{t-1}, x_t)$

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m-history: $x_t^3 = (x_{t-m+1},\ldots.., x_t)$

The correlation integral is defined as:

$$C_m(\varepsilon) = \lim_{N\to\infty} \frac{1}{N^2} \left\{ \text{number of pairs } (i, j) \text{ such that } \| x_i^{\mathrm{m}} - x_j^{\mathrm{m}} \| < \varepsilon \right\}. \quad (5)$$

Intuitively, the correlation integral $C_m(\varepsilon)$ is the fraction of pairs, (x_i^m, x_j^m) which are ε close to each other. For small values of ε, $C_m(\varepsilon)$ grows exponentially at the rate v_m

$$C_m(\varepsilon) \approx \varepsilon^{\upsilon_m}. \quad (6)$$

Calculate the slope of the graph of $\log C_m(\varepsilon)$ versus log ε for small values of ε.

$$v_m = \lim_{\varepsilon\to 0} d \log C_m(\varepsilon) / d \log \varepsilon . \quad (7)$$

If v_m does not increase with *m* and stabilizes at a certain level, this implies that the underlying data generating process is consistent with chaotic behavior: it indicates the presence of a low-dimensional attractor when the estimated correlation dimension stabilizes before *m* reaches 10.

In fact, the Grassberger-Procaccia correlation dimension is defined as

$$v = \lim_{m\to\infty} v_m . \quad (8)$$

However, this approach has its own limitations. First, the correlation dimension methodology lacks an underlying statistical theory: it is basically a graphical analysis that requires very large data sets that are often difficult to obtain in several fields, including economics, and cardiology. Second, small data sets are shown to reduce the correlation dimension estimates, and thus bias the results in favor of finding low dimensional chaos (Ramsey et al., 1990). A new approach to estimating the statistical significance of correlation dimensions is used in the analysis of exchange rate data (Cecen and Erkal, 1996b).

b. The BDSL Test for Temporal Dependence

Another test that has been widely used in the applications of chaos analysis to economic data is the Brock-Dechert-Scheinkman-LeBaron (BDSL) test for temporal dependence (Brock et al., 1996). Using the correlation integral $C_m(\varepsilon)$ defined in (6), the BDSL test statistic is defined as

$$S(m,\varepsilon) = C_m(\varepsilon) - \left[C_1(\varepsilon)\right]^m . \tag{9}$$

The null hypothesis is Ho: x_t is iid. (10)

It is shown that for large samples under the null, $S(m,\varepsilon)$ is asymptotically distributed as normal, i.e.

$$S(m,\varepsilon) \sim N(0,q), \tag{11}$$

where q is a complicated expression depending on sample size, m and ε .

In practice, the BDSL statistic is applied to the residuals of a fitted model as a diagnostic test. First a model is specified and then tested to see if the fitted model is adequate to capture all the dependencies that exist in the data by generating residuals that are iid. It is important to note here that the test is constructed under the null hypothesis of iid, and that rejection of the null does not imply chaos. Nevertheless, if linear dependencies in the data are properly removed - by fitting, for instance, an AR (Autoregressive), MA (Moving Average) or ARMA (Autoregressive Moving Average) model - rejection of the null may be construed as statistical evidence pointing to the existence of nonlinear dependence in the data. The main problem encountered in the interpretation of the BDSL test results is the difficulty of distinguishing between nonlinear stochastic and nonlinear deterministic processes. It is to be noted here that the results of several earlier studies in finance and economics based on the BDSL test were erroneous, given the fact that the BDSL test does not directly say anything about the presence of chaotic dynamics in the systems under investigation.

c. Another Measure of Complexity: Lyapunov Exponents

The Lyapunov exponent is a quantitative measurement of the degree of the sensitivity of the underlying system to initial conditions. It provides a tool to measure how predictable the system is. Therefore, it is used as a means of detecting chaotic behavior. In order to determine the Lyapunov exponents, a small "ball" of initial conditions around a specific point in the state space is formed. Due to the evolution functions, this ball will be stretched or compressed in each direction. This deformation of the ball can be approximated by three numbers, which are known as Lyapunov exponents. In general, the value of the Lyapunov exponent depends on the initial point of the system. The mathematical definition of the Lyapunov exponent $\lambda(x_o)$ is given as

$$\lambda(x_o) \equiv \lim_{n\to\infty} \frac{1}{n} \ln \left| \left. \frac{df^{n}(x)}{dx} \right|_{x_0} \right|,$$

where $\lambda(x_o)$ denotes the average exponential deformation of initially nearby points and f is the function which maps initial point x_0 to a disturbed position. A negative Lyapunov exponent corresponds to an exponential stretching of any small displacement of the initial condition, while a zero Lyapunov exponent indicates a stable displacement. The presence of a positive value, on the other hand, is an indication that any small displacement of the initial condition will grow exponentially over time. This characterizes chaotic behavior. Furthermore, the size of this positive exponent provides a qualitative measurement of how quickly uncertainty about the final state will evolve. Therefore, the sign and size of the Lyapunov exponent together provides a picture of the system in the sense that existence of chaos and the degree of predictability over time can be quantitatively assessed.

Much of earlier empirical work on chaos has relied on the estimation of Lyapunov exponents and erroneously found low dimensional chaos in several economic time series on the basis of positive Lyapunov exponents. It is known, however, that positivity of the largest Lyapunov exponent is not a sufficient condition for chaos: positive Lyapunov exponents can also occur in stochastic systems. In fact, for a given time series, the whole spectrum of Lyapunov exponents must be computed to see if the sum is negative (Eckmann and Ruelle, 1986). On the other hand, the statistical properties of the Lyapunov exponents need to be considered to ascertain the statistical significance of the exponent estimates (Shintani and Linton, 2003).

d. Approximate Entropy as a Measure of Regularity in Time Series Data

The Kolmogorov- Sinai entropy measure is a well - known concept in the theory of complexity. Similarly, Approximate Entropy (ApEn) has been recently introduced into the literature as a new measure of regularity used in clinical studies (Pincus, 1994). It is somewhat akin to the concept of correlation integral, yet it is claimed that it can be computed accurately even with small sample data (Pincus, 1991). The formal definition of ApEn can be summarized as follows. For $r > 0$ and m, a positive integer; form vectors $x_i = [u_i, \ldots, u_{i+m-1}]$ with u_i's generated by a discrete time process $\{U_i\}$. Define the distance $d[x_i, x_j] = \max_{k=1, 2, \ldots, m} \left| u_{i+k-1} - u_{j+k-1} \right|$. Define, for each $i \leq N - m + 1$, $C_i^m(r)$ = (number of $j \leq N - m + 1$ such that $d[x_i, x_j] \leq r$) / (N – m + 1). The $C_i^m(r)$'s measure within a tolerance r the count of patterns similar to a given pattern of window length m. Define $\phi_r^m = (N - m + 1)^{-1} \sum_{i=1}^{N-m+1} \log C_i^m(r)$ and if it exists almost surely, ApEn $(m, r) = \lim_{N \to \infty} [\phi_r^m - \phi_r^{m+1}]$.

Given N data points, we implement this formula by defining the statistic ApEn $(m, r, N) = \phi_r^m - \phi_r^{m+1}$. It follows easily that

$\phi_r^{m+1} - \phi_r^m$ = average of log [conditional probability that $\left| u_{i+m} - u_{j+m} \right| \leq r$, given that $\left| u_{i+k} - u_{j+k} \right| \leq r$ for $k = 0, 1, \ldots, m-1$.

ApEn is therefore aimed at measuring the (logarithmic) likelihood that patterns that are close for m observations remain also close on next incremental comparisons. The higher the "regularity" in the data (that is, the greater the likelihood of remaining close), the smaller will be the ApEn values. Larger ApEn values indicate greater independence, hence greater complexity in the data. Pincus (1994) emphasizes that clinical findings associate sickness and aging with significantly decreased ApEn values. This, in turn, may imply that decreased complexity or greater regularity in the experimental time series are associated with disease and the emergence of a pathological state.

4. Empirical Analysis of Higher-Frequency Exchange Rate Returns

For at least the last two decades, exchange rate dynamics has been the subject of intensive economic research. While earlier work concentrated on exchange rate determination, the recent literature focused attention on the statistical properties of the underlying time series data. The consensus that emerged is rather simple: it is now widely recognized that exchange rate returns display little linear dependence; are characterized by fat-tailed unconditional distributions and changing conditional variance. Hence, exchange rate returns, while exhibiting little correlation, are not temporally independent.

Nonlinearities in exchange rate returns have been extensively scrutinized, among the others, by Hsieh (1989, 1991), Baillie and Bollerslev (1991), and Baillie et al. (2000, 2004). The rise of complex dynamics in economic analysis, however, has shifted attention from stochastic dependence to deterministic dependence in the foreign exchange rate data. With the tools often borrowed from physics, several studies investigated the possibility of a low-dimensional attractor in exchange rate returns (Hsieh, 1989, Bajo-Rubio et al., 1992, and Cecen and Erkal, 1996a, b). While little empirical evidence has been found in favor of chaotic behavior in exchange rate returns, nonlinear dependence in the form of conditional heteroscedasticity in ARCH (Autoregressive Conditional Heteroskedastic) and GARCH (Generalized Autoregressive Conditional Heteroskedastic)-type models – such as Milhoj (1987), Baillie and Bollerslev (1989, 1991), inter alia - was deemed to account for most of the nonlinear effects in the data. Thus, a consensus has emerged that there is indeed little predictability in the exchange rate returns for the nonlinearities work through the conditional variance (or other higher-order conditional moments) rather than via the conditional or unconditional mean. This view seemed consistent with the hypothesis of market efficiency, despite recent challenges that there are serious deviations from efficiency in some asset markets.

In this section we compute the correlation dimension of two sets of exchange rate data. The first set consists of hourly data for the British Pound, Deutsche Mark, and Swiss Franc. We also investigate the nonlinear effects remaining in the residuals from a MA-FIGARCH (Moving Average Fractionally Integrated Generalized Autoregressive Conditional Heteroskedastic) model specification – for the details of the model specification, see Baillie et al. (2000). The data consist of 2,928 observations for hourly and 7,500 observations for 30-minute exchange rate returns. The returns are computed by taking the logarithmic difference of the spot exchange rates multiplied by 100.

Tables 2a–2c and 3a–3c provide the correlation dimension estimates for the hourly and 30-minute exchange rate returns. We report here all the estimates for different delay times, a point often omitted in previous studies. For the two sets of data, we compute the correlation dimensions for embedding dimensions of $M = 1, 2, \ldots, 10$ and for 9 different delay times for hourly and 11 delay times for 30-minute data. The results show that as the embedding dimension M is increases from 1 to 10, the correlation dimension continues to increase: there is no evidence for low dimensional chaos.

Furthermore Table 4 gives the results of the BDSL test for the residuals from the MA-FIGARCH estimated model for the hourly and 30-minute returns. In the correlation dimension analysis, ε is customarily chosen as $\varepsilon = k$*(standard deviation), where $0.25 \leq k \leq 1.25$. Here, we only report the results for $k = ½$ where the embedding dimension M changes from 2 to 10 - the results were similar for other values of k. Clearly all the BDSL statistics consistently fail to reject that the data are iid. We conclude that there is little nonlinearity left in the data sets that are not captured by the MA-FIGARCH specification. This may be construed as evidence that it is stochastic nonlinear dependence, rather than deterministic nonlinear dependence, which characterizes the dynamics of the data generating process.

5. Analysis of the Stock Market Data:

Before the emergence of complex dynamics, technical analysts often made their forecasts on the basis of various linear models, such as capital budgeting, linear regression etc. Capital asset pricing models as well as modern portfolio theory relied, for the most part, on a variety of linear model specifications. While the linear modeling approach assumed a normal (Gaussian) distribution of the random variables of interest, the existence of prolonged and complex stock market phenomena, such as secular bull or bear market episodes over a long time frame appeared to be at odds with this very assumption of thin-tailed distribution.

The movements in stock-market prices are intrinsically stochastic and highly sensitive to the random flow of news and have been construed to be consistent with the efficient market hypothesis: predictability was out of question. Yet a number of significant developments cast doubt on the inherent unpredictability of stock prices. First, econometric research uncovered several serious deviations from the efficient market hypothesis in stock prices (Fama, 1991). Second, statistical analysis of asset returns revealed strong nonlinearities in the data. In light of these developments, chaos theory offers the possibility to uncover a deterministic structure in the seemingly random financial data, hence the possibility of at least short-term prediction in the case a low dimensional attractor.

Much research has been conducted on the question whether financial markets possess an inherently nonlinear structure, whether financial data are characterized by deterministic chaos or some other nonlinear stochastic process. While the earlier studies on nonlinear dynamics in financial markets hinted at the possibility of low dimensional chaos in the stock markets (Scheinkman and LeBaron, 1989) - subsequent research found little evidence in favor of chaotic dynamics in financial data (Abhyankar et al.,1995, 1997), Cecen and Erkal (1996a), Ashley and Patterson (1989), Hsieh (1991), Pandey et al. (1998), and Liu et al. (1992).

Tables 5 and 6 report the results of the correlation dimension estimates for the S&P500 (Standard and Poor's 500) and DJIA (Dow Jones Industrial Averages) indexes respectively. The data involve the daily indexes of S&P for the period January 2 to December 31, 1992 and those of the DJIA for the period of January 2nd to September 17, 1999. Furthermore, the results of the BDSL test are given in Table 7. The results in Tables 5 and 6, computed under various delay times, indicate that as the embedding dimension M increases from 1 to 10, the correlation dimension gradually grows from around 1 to approximately 6. There is no indication that the correlation dimension estimates stabilize around a certain value. Thus, it is possible to conclude that there is no evidence for low dimensional chaotic dynamics in daily S&P 500 and DJIA indexes. Moreover, to test for independence in the data, the BDSL test has been applied and the results are listed in Table 7. The epsilon for S&P 500 is chosen as 0.4264 - which is half the standard deviation 0.8529 of the data. On the other hand, the epsilon for DJIA is chosen as 0.0087 - which is half the standard deviation 0.01745 of the data. The BDSL test statistics appear to indicate that there is indeed temporal dependence in daily stock indexes. Yet in light of our correlation dimension estimates, it is not possible to conclude that this temporal dependency is induced by a deterministic low dimensional attractor. Consequently stochastic nonlinear dependence seems to account for the underlying dynamics.

6. The Analysis of the Heartbeat Data

During the last two decades, recognition that physiologic time series contain ‘‘hidden information’’ has fueled growing interest in applying concepts and techniques borrowed from statistical physics, including chaos theory, to a wide range of biomedical problems from molecular to organismic levels. In the case of cardiology, by assessing the dynamic behavior of arrhythmias, the researchers hoped to uncover the rules that govern their dynamics.

Series of sequential data arise throughout epidemiology in multifaceted contexts. Examples, *inter alia*, include (i) hormonal secretory dynamics based on frequent, fixed increment samples from serum; (ii) electrocardiographic (ECG) and heart rate time-series; (iii) electroencephalograms (EEGs). The ability to quantify the intrinsic differences among such series is extremely valuable, since in their respective contexts, these series contain essential biological information. Although practitioners and researchers typically quantify mean levels or the extent of variability, it is now recognized that in many instances, the persistence of certain patterns, or shifts in an 'apparent ensemble amount of randomness', may provide important new insights into the subject status (Kaplan and Talajic 1991), Pikkujamsa et al. (1999), and Ivanov et al. (2001).

This section focuses on the quantification of randomness in the heartbeat data using Approximate Entropy. Approximate Entropy (ApEn), a measure recently introduced into the literature by Pincus (1991), is aimed at quantifying the irregularity (hence entropy) in a time series data. It is important to note here a fundamental difference between regularity statistics, such as ApEn, and other variability measures: for variability measures, the order of the input data is irrelevant - the focus is to quantify the degree of spread about a central value. In contrast, for ApEn, discerning changes in order from apparently random to very regular is the primary statistical focus. ApEn has been applied to numerous settings both within and outside biology. In heart rate studies, ApEn has revealed very significant differences in a variety of contexts in which traditional moment statistics (such as the mean and the variance of the data) help identify little clear distinction.

Recently, ApEn has been shown to be progressively decreasing starting from 120 min. prior to the atrial fibrillation episodes. Furthermore, ApEn has proved to be helpful in risk assessment of patients who had myocardial infraction.

Thus by analyzing the entropy of the beat-beat intervals of cardiac cycles, it has been possible to risk stratify certain patient populations. The decrease in entropy of the heart rate time series data has been shown to be associated with decreased survival or worsened prognosis in some disease states. This entropy change of the beat-to-beat intervals seems to be related to the altered control mechanisms that originate from the brain and regulate the cardiac rhythm.

Atrial fibrillation (AF) is the most common rhythm disorder in adults. The first set of data is RR interval (beat-to-beat intervals in msec) data belonging to a patient with this rhythm disorder. During an AF episode, there is a very fast irregular rhythm in the upper chambers of the heart. RR intervals are the product of conduction of this irregular rhythm through the atrio-ventricular node (the electrical conduction center present between the upper and lower chambers of the heart). The second set of data is the heartbeat data of a normal (no disease) person. The correlation dimension estimates of these two stationary series are presented in Tables 8 and 9. The interpretation of the correlation dimension estimates is far from being clear. While there is some evidence that the first set of estimates (with AF) fails to saturate as the embedding dimension increases across all delay times, the correlation dimension estimates belonging to a normal individual appear to saturate at delay times 2, 3 and 8. In fact some studies have claimed that normal heart rhyhms showed some features of deterministic chaos – for a discussion of chaotic dynamics in heartbeat data, see Lefebvre et al. (1993).

Tables 10 and 11 provide the ApEn estimates for the same two data sets for different values of the window length m {2, 3, ..,12} and the tolerance or filter level r $\{\sigma,\ 0.6\sigma,\ 0.2\sigma\}$, where σ is the standard deviation of the data set used, corresponding to e values in the neighborhood of {0.14, 0.085, 0.028} respectively. Here e is the ratio $r/\bar{x}$, where $\bar{x}$ is the mean of the data set used. Different sample sizes and parameter choices are used in order to see the sensitivity of the ApEn estimates to the sample size: they differ considerably from each other, although it is not possible to say anything definitive about the statistical significance of the differences between these estimates since the statistical distribution of ApEn is unknown. Nevertheless the results seem to demonstrate that during an atrial fibrillation episode, entropy estimates are much higher than those in normal times.

7. Concluding Remarks

This study has provided a brief review of the empirics of complex dynamic analysis in order to highlight several critical points pertaining to the tests customarily used in applied research. To illustrate the difficulties and ambiguities associated with the interpretation of the test results, three examples from economics, finance and cardiology are discussed: high frequency exchange rate returns; the S&P 500 and the DJIA stock indices; and the heartbeat data in the case of a healthy individual and a patient inflicted with atrial fibrillation. In light of these three examples, it is possible to emphasize a number of points.

First, generally speaking, all the test results appear to be quite sensitive to sample size as well as to the presence of linear dependence in the raw data. In the case of correlation dimension calculations, large data sets with ideally little noise are required to test for low dimensional chaos. Second, it is equally important to make sure that the time series under investigation are stationary and that linear dependence presented in the data is removed before most of the tests are applied to the time series. For correlation dimension calculations, it is also necessary to consider several delay times. When these test conditions are properly met, contrary to the findings of the earlier literature, it is clear that there is little evidence in favor of low dimensional chaos in economic/financial time series: nonlinear stochastic time dependence seems to account for much of the temporal dependence in these series.

Insofar as cardiology studies are concerned, evidence for low dimensional chaos in heartbeat data is weak at best. The ApEn estimates, on the other hand, appear to be too sensitive to the sample size. While we have only focused on data from two representative subjects randomly selected from a larger sample, it appears that our estimation results are far from corraborating several theoretical conjectures of the earlier literature.

TABLES

Table 1. Autocorrelation (ACF) and Partial Autocorrelation Function (PACF) for Tent and Logistic maps.

	Tent Map		Logistic Map	
Lag	ACF	PACF	ACF	PACF
1	0.007	0.007	0.008	0.008
2	0.017	0.017	-0.004	-0.004
	-		-	
3	0.015	-0.015	0.043	-0.043
4	-0.020	-0.020	0.006	0.007
5	0.001	0.000	0.012	0.012
6	0.003	0.004	0.051	0.049
7	-0.051	-0.051	-0.024	-0.025
8	0.006	0.006	-0.017	-0.015
9	0.021	0.023	-0.012	-0.007
10	-0.030	-0.032	0.028	0.025

Table 2a. Correlation Dimension Estimates for Hourly $/BP Returns.

N / *M*	1	2	4	8	16	32	64	128	256
1	1.008	1.008	1.008	1.007	1.007	1.007	1.004	1.005	1.161
2	1.684	1.841	1.852	1.855	1.878	1.864	1.860	1.869	2.100
3	2.966	2.983	2.996	2.943	2.978	2.959	3.014	3.021	3.068
4	3.819	3.834	3.887	3.877	3.783	3.906	3.884	3.819	4.788
5	4.417	4.445	4.545	4.582	4.493	4.583	4.550	4.593	4.875
6	4.884	4.905	5.049	5.078	5.042	5.142	5.031	5.177	5.477
7	5.294	5.297	5.566	5.553	5.549	5.540	5.556	5.494	6.508
8	5.633	5.614	5.882	5.859	5.827	5.855	5.862	5.907	6.111
9	5.825	5.852	6.149	6.103	6.182	6.117	6.111	6.134	6.774
10	6.172	6.211	6.493	6.480	6.458	6.388	6.438	6.511	6.605

Notes: *N* is Delay Time. *M* is Embedding Dimension.

Table 2b. Correlation Dimension Estimates for Hourly DM/$ Returns.

N / *M*	1	2	4	8	16	32	64	128	256
1	1.027	1.027	1.027	1.027	1.026	1.028	1.032	1.036	1.066
2	1.885	2.021	2.017	1.996	2.007	2.027	1.983	2.043	1.991
3	2.978	2.986	3.034	3.001	2.992	3.035	2.984	3.038	3.231
4	3.808	3.824	3.886	3.876	3.805	3.927	3.878	3.951	3.974
5	4.426	4.550	4.523	4.563	4.566	4.571	4.586	4.541	4.615
6	4.877	5.027	4.994	5.041	5.043	5.100	5.145	5.129	4.777
7	5.321	5.510	5.516	5.548	5.443	5.567	5.612	5.738	5.826
8	5.745	5.766	5.801	5.878	5.818	5.853	5.770	5.785	5.863
9	5.846	6.015	6.066	6.174	6.082	6.135	6.135	6.202	6.279
10	6.182	6.338	6.408	6.534	6.427	6.407	6.397	6.605	6.059

Notes: *N* is Delay Time. *M* is Embedding Dimension.

Table 2c. Correlation Dimension Estimates for Hourly SF/$ Returns.

N / *M*	1	2	4	8	16	32	64	128	256
1	1.026	1.026	1.025	1.025	1.026	1.026	1.017	1.002	0.928
2	1.860	2.004	2.010	2.021	2.025	2.029	1.992	2.085	1.988
3	3.005	3.000	3.004	2.964	2.950	2.997	3.028	3.019	3.070
4	3.763	3.805	3.812	3.773	3.763	3.804	3.822	3.860	3.987
5	4.315	4.430	4.428	4.363	4.380	4.408	4.429	4.477	4.355
6	4.751	4.823	4.851	4.783	4.732	4.895	4.847	4.939	4.558
7	5.109	5.240	5.304	5.238	5.182	5.422	5.284	5.528	6.880
8	5.394	5.562	5.630	5.524	5.573	5.776	5.563	5.747	5.664
9	5.682	5.907	5.875	5.736	5.885	6.108	5.785	6.158	6.044
10	6.039	6.189	6.269	6.129	6.175	6.447	6.072	6.338	6.205

Notes: *N* is Delay Time. *M* is Embedding Dimension.

Table 3a. Correlation Dimension Estimates for 30 minute $/BP Returns.

N / *M*	1	2	4	8	16	32	64	128	256	512	1024
1	1.027	1.027	1.027	1.027	1.027	1.027	1.027	1.027	1.027	1.026	0.991
2	2.050	2.068	2.073	2.066	2.067	2.066	2.077	2.069	2.076	2.075	2.111
3	3.009	3.036	3.043	3.037	3.039	3.049	3.045	3.044	3.035	3.023	3.067
4	3.827	3.848	3.841	3.841	3.843	3.846	3.833	3.853	3.849	3.844	3.877
5	4.561	4.596	4.592	4.581	4.580	4.599	4.576	4.603	4.610	4.607	4.511
6	5.073	5.112	5.108	5.113	5.102	5.125	5.113	5.121	5.107	5.109	5.151
7	5.486	5.537	5.553	5.557	5.539	5.555	5.546	5.560	5.555	5.536	5.564
8	5.820	5.887	5.909	5.929	5.895	5.914	5.904	5.921	5.910	5.893	6.013
9	6.185	6.235	6.281	6.325	6.273	6.319	6.290	6.296	6.285	6.293	6.473
10	6.254	6.273	6.303	6.330	6.286	6.290	6.302	6.307	6.286	6.281	6.343

Notes: *N* is Delay Time, *M* is Embedding Dimension.

Table 3b.Correlation Dimension Estimates for 30 minute DM/$ Returns.

N / *M*	1	2	4	8	16	32	64	128	256	512	1024
1	1.029	1.029	1.029	1.029	1.029	1.030	1.030	1.030	1.030	1.030	1.008
2	2.057	2.063	2.065	2.064	2.069	2.076	2.069	2.068	2.069	2.074	2.092
3	3.044	3.042	3.035	3.037	3.048	3.049	3.045	3.042	3.044	3.052	3.041
4	3.839	3.824	3.838	3.844	3.858	3.856	3.857	3.857	3.850	3.856	3.818
5	4.496	4.561	4.570	4.575	4.617	4.607	4.539	4.610	4.598	4.601	4.533
6	5.081	5.078	5.122	5.116	5.131	5.129	5.144	5.145	5.153	5.129	5.244
7	5.501	5.510	5.570	5.557	5.568	5.563	5.585	5.567	5.591	5.587	5.633
8	5.861	5.892	5.948	5.944	5.932	5.949	5.941	5.942	5.949	5.946	5.906
9	6.181	6.268	6.333	6.333	6.302	6.332	6.313	6.345	6.331	6.336	6.230
10	6.288	6.320	6.345	6.324	6.319	6.331	6.334	6.329	6.357	6.358	6.349

Notes: *N* is Delay Time, *M* is Embedding Dimension.

Table 3c. Correlation Dimension Estimates for 30 minute SF/$ Returns.

N	1	2	4	8	16	32	64	128	256	512	1024
M											
1	1.024	1.024	1.024	1.024	1.024	1.025	1.025	1.025	1.025	1.023	1.018
2	2.043	2.057	2.059	2.072	2.067	2.065	2.073	2.058	2.065	2.066	2.068
3	2.998	3.004	3.021	3.024	3.020	3.029	3.043	3.035	3.026	3.032	3.017
4	3.788	3.805	3.824	3.820	3.835	3.824	3.820	3.823	3.826	3.863	3.817
5	4.465	4.529	4.524	4.529	4.543	4.596	4.566	4.574	4.564	4.583	4.480
6	4.956	5.027	5.038	5.035	5.052	5.084	5.097	5.055	5.069	5.067	5.143
7	5.355	5.450	5.464	5.469	5.484	5.518	5.527	5.487	5.515	5.516	5.583
8	5.720	5.808	5.823	5.848	5.834	5.871	5.900	5.867	5.868	5.866	5.833
9	6.048	6.203	6.206	6.232	6.224	6.250	6.297	6.255	6.259	6.246	6.174
10	6.125	6.208	6.245	6.229	6.218	6.288	6.302	6.265	6.279	6.280	6.277

Notes: *N* is Delay Time, *M* is Embedding Dimension.

Table 4. BDSL Statistics for the residuals from the MA(1)–FIGARCH (1, δ, 1) with hourly and 30–minute Returns data.

	Hourly Returns data			**30–minute Returns data**		
M	DM/$	$/BP	SF/$	DM/$	$/BP	SF/$
2	-0.062	-0.073	-0.072	-0.149	-0.066	-0.101
3	-0.083	-0.098	-0.097	-0.163	-0.088	-0.127
4	-0.099	-0.117	-0.122	-0.189	-0.105	-0.140
5	-0.112	-0.129	-0.140	-0.215	-0.120	-0.158
6	-0.125	-0.136	-0.155	-0.237	-0.133	-0.180
7	-0.136	-0.143	-0.168	-0.258	-0.145	-0.199
8	-0.146	-0.151	-0.180	-0.277	-0.157	-0.215
9	-0.156	-0.160	-0.192	-0.296	-0.169	-0.232
10	-0.165	-0.170	-0.201	-0.314	-0.179	-0.247

Notes: For all series, ε = ½*(standard deviation); *M* is Embedding Dimension.

Table 5. Correlation Dimension Estimates for the Daily S&P 500.

N	1	2	4	8	16	32	64	128
M								
1	1.029	1.029	1.029	1.027	1.025	1.025	1.029	1.029
2	2.065	2.085	2.068	2.047	2.066	2.066	2.079	2.067
3	3.021	3.055	3.040	3.055	3.054	3.042	3.094	3.095
4	3.834	3.905	3.895	3.875	3.869	3.933	3.921	3.900
5	4.566	4.625	4.488	4.555	4.629	4.599	4.597	4.603
6	5.069	5.121	5.118	5.075	5.098	5.099	5.196	5.163
7	5.578	5.619	5.612	5.589	5.628	5.578	5.675	5.619
8	5.897	6.083	6.072	6.054	5.905	5.904	5.891	5.758
9	6.150	6.219	6.113	6.127	6.175	6.200	6.253	6.139
10	6.472	6.562	6.400	6.469	6.470	6.541	6.515	6.508

Notes: *N* is Delay Time, *M* is Embedding Dimension.

Table 6. Correlation Dimension Estimates for the Daily DJIA.

N	1	2	4	8	16	32	64	128
M								
1	1.029	1.029	1.029	1.029	1.028	1.029	1.027	1.026
2	2.065	2.055	2.060	2.055	2.065	2.055	2.074	2.070
3	3.054	3.062	3.040	3.014	3.051	3.031	3.035	3.003
4	3.886	3.905	3.904	3.877	3.903	3.878	3.842	3.900
5	4.571	4.581	4.571	4.536	4.594	4.532	4.522	4.579
6	5.049	5.075	5.057	5.050	5.066	5.014	5.103	5.093
7	5.544	5.531	5.511	5.536	5.532	5.447	5.609	5.560
8	6.009	6.001	5.972	5.966	5.973	5.847	6.085	5.962
9	6.013	6.018	5.990	6.056	6.059	5.975	6.089	6.120
10	6.326	6.344	6.308	6.374	6.377	6.266	6.439	6.420

Notes: *N* is Delay Time, *M* is Embedding Dimension.

Table 7. BDSL Statistics for the S&P 500 and DJIA.

	S&P 500	DJIA
M		
2	0.6311	4.2563
3	1.0146	5.8091
4	1.8146	7.2379
5	2.6255	9.1111
6	3.4979	10.900
7	2.0358	12.675
8	-2.8303	14.510
9	-5.9534	16.665
10	-5.8356	18.946

Notes: *M* is Embedding Dimension.

Table 8. Correlation Dimension Estimates for the Atrial Fibrillation (AF) Heartbeat Data.

N	1	2	4	8	16	32
M						
2	0.834	2.180	2.208	2.201	2.217	2.180
3	2.468	2.933	2.966	2.965	2.933	2.963
4	2.974	3.577	3.618	3.585	3.553	3.661
5	3.648	4.092	4.164	4.206	4.117	4.252
6	4.297	4.621	4.710	4.732	4.684	4.779
7	4.893	5.178	5.305	5.347	5.254	5.373
8	5.370	5.635	5.725	5.724	5.754	5.799
9	5.774	5.981	6.090	6.086	6.116	6.081
10	6.098	6.312	6.425	6.400	6.416	6.443

Notes: *N* is Delay Time, *M* is Embedding Dimension.

Table 9. Correlation Dimension Estimates for the Normal (no disease) Heartbeat Data.

N	1	2	4	8	16	32
M						
2	NA	NA	NA	4.443	4.471	4.481
3	4.837	4.929	4.341	4.445	4.021	4.044
4	4.933	5.041	4.313	4.555	4.635	4.004
5	5.407	4.813	5.090	4.528	4.695	4.543
6	5.175	5.377	4.781	4.984	5.008	5.100
7	5.638	4.880	5.069	5.347	5.411	5.510
8	5.599	5.389	5.185	5.579	5.610	5.613
9	5.479	5.403	5.256	5.633	5.747	6.035
10	5.327	5.269	5.649	5.641	6.101	6.374

Notes: *N* is Delay Time, *M* is Embedding Dimension.

Table 10a. Approximate Entropy Estimates during an AF episode.

M	r = 91.6 (e = 0.141)	r = 55 (e = 0.085)	r = 18.3 (e = 0.028)
2	0.745	1.138	1.859
3	0.742	1.114	1.302
4	0.737	1.062	0.497
5	0.730	0.941	0.138
6	0.714	0.694	0.030
7	0.674	0.368	0.006
8	0.609	0.171	0.003
9	0.512	0.068	0.001
10	0.381	0.030	0.001
11	0.260	0.011	0.001
12	0.159	0.004	0.001
	Size: 2, 844	Mean: 650.6	Std. Dev: 91.64

Table 10b. Approximate Entropy Estimates during an AF episode.

M	r = 91.6 (e = 0.141)	r = 55 (e = 0.085)	r = 18.3 (e = 0.028)
2	0.721	1.049	1.678
3	0.716	1.019	0.921
4	0.704	0.960	0.275
5	0.684	0.795	0.046
6	0.660	0.529	0.010
7	0.596	0.249	0
8	0.502	0.118	0
9	0.362	0.033	0
10	0.228	0.010	0
11	0.138	0.003	0
12	0.071	0	0
	Size:1, 000	Mean: 653.8	Std. Dev : 89.63

Table 11a. Approximate Entropy Estimates for a Normal Individual.

M	r = 100 (e = 0.14)	r = 60 (e = 0.085)	r = 20 (e = 0.028)
2	0.124	0.223	0.857
3	0.117	0.216	0.812
4	0.110	0.207	0.739
5	0.106	0.202	0.662
6	0.104	0.199	0.574
7	0.102	0.196	0.492
8	0.101	0.192	0.415
9	0.099	0.187	0.347
10	0.097	0.182	0.289
11	0.094	0.177	0.239
12	0.092	0.171	0.198
	Size: 16,360	Mean: 712	Std. Dev. : 99.8

Table 11b. Approximate Entropy Estimates for a Normal Individual.

M	r = 100 (e = 0.14)	r = 60 (e = 0.082)	r = 20 (e = 0.027)
2	0.068	0.160	0.671
3	0.064	0.154	0.558
4	0.060	0.150	0.462
5	0.057	0.147	0.365
6	0.056	0.143	0.289
7	0.054	0.145	0.232
8	0.054	0.145	0.185
9	0.053	0.136	0.128
10	0.051	0.136	0.091
11	0.050	0.122	0.063
12	0.050	0.119	0.051
	Size:500	Mean: 734	Std. Dev. : 83.2

Table 11c. Approximate Entropy Estimates for a Normal Individual.

M	r = 100 (e = 0.14)	r = 60 (e = 0.084)	r = 20 (e = 0.027)
2	0.045	0.113	0.616
3	0.042	0.108	0.572
4	0.039	0.104	0.504
5	0.037	0.102	0.453
6	0.037	0.100	0.409
7	0.037	0.101	0.360
8	0.037	0.101	0.303
9	0.036	0.096	0.265
10	0.035	0.096	0.217
11	0.034	0.089	0.174
12	0.034	0.087	0.132
	Size: 1, 000	Mean: 700	Std. Dev. : 70.7

Table 11d. Approximate Entropy Estimates for a Normal Individual.

M	*r* = 100 (*e* = 0.14)	*r* = 60 (*e* = 0.083)	*r* = 20 (*e* = 0.028)
2	0.108	0.223	0.831
3	0.104	0.216	0.730
4	0.095	0.204	0.573
5	0.089	0.196	0.456
6	0.086	0.192	0.350
7	0.085	0.187	0.281
8	0.084	0.185	0.230
9	0.083	0.180	0.201
10	0.081	0.174	0.166
11	0.079	0.164	0.135
12	0.077	0.152	0.104
	Size: 2, 000	Mean: 720.6	Std. Dev.: 82.5

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