4 Isomorphism

4.1 Bijection

We recall several definitions concerning mappings between sets. Let $A$ and $B$ be any two sets and $f : A \rightarrow B$ be any mapping (note that the terms map, mapping and function have the same meaning).

A mapping $f : A \rightarrow B$ is called one-to-one (or injective) if each element of $B$ has at most one element of $A$ mapped into it. In other words, $f$ is one-to-one, if for any two distinct points $x_1$ and $x_2$, the points $f(x_1)$ and $f(x_2)$ are also distinct ($f$ sends different points into different points).

Example 4.1. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that for every $x$ the derivative $f'(x)$ exists and is positive. Then $f$ is one-to-one, since for any $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$, one has

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x) dx > 0,$$

and thus $f(x_1) \neq f(x_2)$.

A function $f : A \rightarrow B$ is onto $B$ (or surjective), if each element of $B$ has at least one element of $A$ mapped into it. Finally, a function $f : A \rightarrow B$ is a bijection between $A$ and $B$, if it is both one-to one and onto $B$.

For any bijection $f : A \rightarrow B$, one can define the inverse $f^{-1} : B \rightarrow A$ such that $f^{-1}(f(x)) = x$ for any $x \in A$ and $f(f^{-1}(y)) = y$ for any $y \in B$. The inverse function $f^{-1}$ is bijective as well as $f$.

4.2 Definition and examples

Example 4.2. Consider the group $U_3$. Denote $a = e^{2\pi i/3}$, $b = e^{4\pi i/3}$. The Cayley table for $U_3$ is

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>$a$</th>
<th>$b$</th>
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</thead>
<tbody>
<tr>
<td>1</td>
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<td>$a$</td>
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<td>$a$</td>
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<tr>
<td>$b$</td>
<td>$b$</td>
<td>1</td>
<td>$a$</td>
</tr>
</tbody>
</table>

Next, consider the group $S_3$. Denote $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. The set $H = \{i, \alpha, \beta\}$ is a subgroup of $G$. The Cayley table for $H$ is

<table>
<thead>
<tr>
<th></th>
<th>$i$</th>
<th>$\alpha$</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i$</td>
<td>$i$</td>
<td>$\alpha$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\alpha$</td>
<td>$\beta$</td>
<td>$i$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$\beta$</td>
<td>$i$</td>
<td>$\alpha$</td>
</tr>
</tbody>
</table>
Notice that the two tables are identical up to notation.

In the above example, the two groups appear to be essentially the same. What does the word ‘essentially’ mean here? The definition of isomorphism below makes this precise.

**Definition 4.3.** Let $G$ and $H$ be two groups. A function $f : G \to H$ is called an **isomorphism** between $G$ and $H$, if

(i) $f$ is a bijection;
(ii) for any $a, b \in G$, one has $f(ab) = f(a)f(b)$.

If there exists an isomorphism between two groups, then the groups are called isomorphic (to each other).

Thus, an isomorphism $f$ gives a one-to-one correspondence between the elements of two groups, which preserves the group operation. In other words, the fact that two groups are isomorphic, means that we can simply rename the elements of the first group (calling $x$ by $f(x)$) and obtain exactly the second group. Obviously, the way we name elements is not important, so from the point of view of the group theory, isomorphic groups are identical.

**Example 4.4.** In the Example 4.2, let $G = U_3$ and consider the mapping $f : G \to H$, defined by $f(1) = \iota$, $f(a) = \alpha$, $f(b) = \beta$. Then $f$ is an isomorphism between $G$ and $H$.

The following theorem shows that an isomorphism preserves not only products, but also the identity element and inverses.

**Theorem 4.5.** Let $G$ and $H$ be two groups, and $f : G \to H$ be an isomorphism between $G$ and $H$.

(i) If $e$ is the identity element in $G$, then $f(e)$ is the identity $\bar{e}$ in $H$.

(ii) If $g \in G$ and $g^{-1}$ is the inverse of $g$, then the inverse $(f(g))^{-1}$ of $f(g)$ is $f(g^{-1})$.

**Proof.** (i) For any $g \in G$, one has

$$f(g)\bar{e} = f(g) = f(ge) = f(g)f(e).$$

Using the cancellation rule, we obtain $\bar{e} = f(e)$.

(ii) Since $gg^{-1} = g^{-1}g = e$, one has

$$\bar{e} = f(e) = f(g^{-1}g) = f(g^{-1})f(g),$$

$$\bar{e} = f(e) = f(gg^{-1}) = f(g)f(g^{-1}).$$

By Theorem 2.3(ii) (uniqueness of the inverse), $f(g^{-1})$ must be the inverse of $f(g)$. 

**Example 4.6.** Consider the cyclic subgroup $H$ of $S_4$, generated by the element $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$. Clearly, $H$ consists only of $\sigma$ and the identity $\iota$. It is easy to check that $H$ is isomorphic to $U_2$. 

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Example 4.7. Let us show that \( \mathbb{R} \) under addition is isomorphic to \( \mathbb{R}^+ \) under multiplication.

Define the mapping \( f : \mathbb{R} \to \mathbb{R}^+ \) by \( f(x) = e^x \). Since \( f'(x) = e^x \) for every \( x \in \mathbb{R} \), this mapping is one-to-one. Since

\[
\lim_{x \to -\infty} e^x = 0, \quad \lim_{x \to \infty} e^x = \infty,
\]

the range of \( f \) is \((0, \infty)\), i.e., \( f \) is onto. Thus, \( f \) is a bijection.

Next, for any \( x, y \in \mathbb{R} \),

\[
f(x + y) = e^{x+y} = e^x e^y = f(x)f(y),
\]

and so \( f \) is an isomorphism.

Group theory studies the properties of the groups that are shared by any isomorphic group. Such properties are called structural properties. Let us give examples of structural properties of a group:

- The group is Abelian.
- The number of elements of the group is 10.
- The group has a finite number of elements.
- The equation \( x^2 = a \) has a solution for each element \( a \) of the group.
- There are exactly two elements \( a \) in the group, satisfying the equation \( a^2 = e \) (where \( e \) is the identity).
- The group has two cyclic subgroups
- The group has no non-trivial proper subgroups

As we progress in our study of groups, we will see a lot of more interesting structural properties.

Let us also give examples of non-structural properties of a group:

- The group contains 5.
- The group operation is denoted by \( * \).
- All elements of the group are numbers.
- The group contains no matrices.
- The group is specified by its Cayley table.

**How to show that two groups are isomorphic?**

If \( G \) and \( H \) are the groups in question, you need to construct an isomorphism \( f : G \to H \).

This is usually done as follows:
1. Define a function \( f : G \rightarrow H \), which you think will be an isomorphism.

2. Prove that \( f \) is a one-to-one function.

3. Prove that \( f \) is onto.

4. Prove that \( f(xy) = f(x)f(y) \) for all \( x, y \in G \).

**How to show that two groups are not isomorphic?**

If two groups are isomorphic, all their structural properties are the same. Thus, if one of the groups has some structural property, while the other does not, then these groups are not isomorphic. Let us give a couple of examples of this situation.

**Example 4.8.**

1. The group \( U_3 \) is not isomorphic to \( U_4 \), because \( U_3 \) has 3 elements, and \( U_4 \) has 4 elements.

2. The group \( \mathbb{R}^* \) under multiplication and the group \( \mathbb{C}^* \) under multiplication are not isomorphic. The equation \( x^2 = a \) has solution \( x \in \mathbb{C}^* \) for any \( a \in \mathbb{C}^* \), but the same equation does not have a solution in \( \mathbb{R}^* \) for some \( a \in \mathbb{R}^* \) (namely, for negative \( a \)).

3. The group \( \mathbb{R} \) under addition is not isomorphic to the group \( \mathbb{R}^* \) under multiplication. The equation \( x^2 = 1 \) in \( \mathbb{R}^* \) has two solutions (1 and \(-1\)), and the corresponding equation \( x + x = 0 \) in \( \mathbb{R} \) has only one solution 0.

**Remark.** Note that the ideas discussed in this section are applicable to many branches of mathematics. Most axiomatic theories study sets with some additional structure. In our case this is a group structure. Another example is a linear structure for linear algebra. In Euclidean geometry, such structure is distances between points and angles between lines of the geometric figures.

Isomorphisms in any such theory are bijections which preserve the additional structure. Isomorphism is always an equivalence relation. Isomorphic objects have the same structural properties and may differ only in the explicit nature of their elements. The word ‘structural’ here is understood with respect to a particular theory.